A Theory of Multiperiod Debt Structure *

Chong Huang
UC Irvine

Martin Oehmke
London School of Economics

Hongda Zhong
London School of Economics

Abstract

We develop a theory of multiperiod debt structure. A simple trade-off between the termination threat required to make debt repayments incentive compatible and the desire to avoid early liquidation determines the number of repayments, their timing, and amounts. As firms increase their borrowing, they add periodic risky repayments from the back of the maturity structure, with the time between repayments increasing in cash-flow risk. Cash-flow growth or a significant risk-free cash-flow component limits the number of risky repayments. Firms with significant risk-free cash-flow component choose dispersed maturity profiles with smaller, relatively safe repayments every period, rather than riskier periodic repayments. (*JEL* G30, G32, G33)

*For comments and suggestions, we thank two anonymous referees and Ulf Axelson, Bruno Biais, Patrick Bolton, Philip Bond, Mike Burkart, Ing-Haw Cheng, Vicente Cuñat, Jesse Davis, Theodosios Dimopoulos, Itay Goldstein, Denis Gromb, Zhiguo He, David Hirshleifer, Dirk Jenter, Yan Ji, Konstantin Milbradt, Emilio Osambela, Giorgia Piacentino, Anatoli Segura, Stijn Van Nieuwerburgh, Sergio Vicente, Lucy White, and Jing Zeng. We are also grateful to seminar and conference participants at LSE, UC Irvine, the London FIT workshop, Einaudi Institute, the WBS Frontiers of Finance Conference, the 6th ITAM Finance Conference, the Barcelona GSE Summer Forum, CICF, SED, ESSFM Gerzensee, BI Norwegian Business School, New Economic School, ICEF Moscow, Stockholm School of Economics, the HKUST Finance Symposium, the Colorado Finance Summit, the 2017 GEA conference at the Bundesbank, the 2018 AFA, the 2018 RCFS conference, the 2018 FIRS conference, WU Vienna, the 2018 EFA, Bocconi, and Bristol. We thank Ran Shi for excellent research assistance. M. O. gratefully acknowledges funding from the ERC [starting grant 714567]. Send correspondence to Martin Oehmke, Department of Finance, London School of Economics, Houghton Street, London WC2A 2AE, UK. E-mail: m.oehmke@lse.ac.uk.

How do firms choose the term structure of their debt? While a large literature has investigated why firms use debt to raise financing for investments,[1] we know far less about the determinants of the number of repayment dates, their timing, and their respective repayment amounts. To shed light on these issues, this paper develops a model of multiperiod debt financing in a setting with privately observed cash flow. In our model, a rich term structure of debt emerges from a simple trade-off between providing the firm with incentives to repay and preventing costly early liquidation.

The main friction in our model draws on classic models of debt financing: Cash flow is privately observed by the entrepreneur, such that the entrepreneur can abscond with the cash flow instead of repaying debt. Therefore, like in Bolton and Scharfstein (1990, 1996) and Hart and Moore (1998), debt induces a termination threat that makes repayment incentive compatible. However, in contrast to the two-period nature of these papers, in our model the firm produces cash flow over many periods. This multiperiod setup allows us to study the optimal debt structure: How many repayment dates should there be? What should the timing and size of those repayments be?

The key trade-off that determines the optimal debt structure balances default risk with the incentives necessary to ensure repayment. Repayment incentives derive from the threat of early termination. Specifically, the firm's creditors commit to liquidating the firm if it defaults on any of its contractual repayments. Early liquidation is costly because it leads to the loss of a longer stream of future cash flows. The entrepreneur therefore would like to schedule debt repayments as late as possible. However, there is a limit to how late repayments can be credibly made to creditors: toward the end of the project, the entrepreneur's continuation value is lower, leading to larger incentives to divert the cash flow and default.

In our baseline model, the firm generates a risky cash flow every period, independently drawn from the same binary distribution (zero or positive). In this setting, we show that a repayment

[1] Classic contributions to this literature include models of costly state verification (Townsend (1979); Gale and Hellwig (1985)), termination threat models of debt (Bolton and Scharfstein (1990, 1996); Hart and Moore (1994, 1998)), incentive-based theories of debt (Innes 1990), and theories based on information sensitivity (Gorton and Pennacchi (1990); Dang, Gorton, and Holmström (2015)).

profile with evenly spaced debt payments toward the end of the project is optimal, where the time between repayment dates is determined by the riskiness of the firm's cash flows. On each repayment date, the firm pays back the entire realized period cash flow, in order to minimize the number of risky payments. The cash flows between payment dates accrue to the entrepreneur, thereby providing incentives to honor each of the contractual payments. All else equal, the larger the amount of outside financing that the firm needs to raise, the larger the number of repayment dates and the more front-loaded the repayment schedule. A key feature of our baseline model is that pledgeable income is maximized by scheduling as many repayments as possible, subject to spacing these repayments such that they are incentive compatible. As a consequence, firms with large outside financing needs opt for more repayment dates and earlier average repayment times. This result echoes and extends the classic insight (Bolton and Scharfstein (1990, 1996) and Hart and Moore (1998)) that short-term debt alleviates financing constraints that arise under incomplete contracting.

We then extend our model to more general cash-flow distributions. Interestingly, the result that a large number of repayment dates maximizes pledgeable income no longer holds when the firm's cash flow grows over time or has a significant risk-free component. When there is growth in the firm's expected cash flows, pledgeable income is generally maximized by a debt contract with relatively few risky repayments toward the end of the project's life (even though the firm could in principle add more repayment dates). In fact, in some cases a single (bullet) repayment maximizes pledgeability. For growth firms, the optimal debt structure therefore resembles long-term debt with debt maturity closer to the maturity of the firm's assets.

When the firm generates a positive minimum cash flow in each period, the debt contract that maximizes pledgeable income depends on the riskiness of the firm. When the safe cash-flow component is large relative to total cash flow, pledgeability is maximized by offering a safe repayment in every period. If, on the other hand, the risky part of the cash flow makes up a significant fraction of the firm's overall cash flow, pledgeability is maximized by alternating between safe and risky repayments. While safe repayments occur throughout the lifetime of the firm's assets, risky repayments are scheduled toward the end of the project and need to be appropriately spaced to preserve incentive compatibility. Like the case with cash-flow growth, also in this case increasing the number of risky payments only raises pledgeability up to a point: pledgeability is generally maximized with a fixed number of risky repayments that is independent of project maturity. In this case, the optimal debt structure resembles a combination of safe repayments and a number of risky long-term bonds or loans.

The intuition gained from the model in which the binary cash flow has a risk-free component carries over to continuous distributions. Specifically, we numerically analyze the optimal debt structure when the period cash flow follows a lognormal distribution. The analysis shows that when period cash flows have limited upside (corresponding to low volatility), the firm's debt structure features frequent small and relatively safe repayments in every period. In contrast, when period cash flows have significant upside (corresponding to high volatility), the firm's debt structure is lumpy, with less frequent but larger repayments, spaced out to guarantee incentive compatibility. Our model therefore provides a unified framework that can capture incentives to finance using both a smooth debt structure, consisting of frequent and less risky repayments, and a lumpy debt structure, in which repayments are more infrequent and riskier.

Our model provides a unified framework to explain the choice of average debt maturities as well as the granularity of corporate debt (i.e., how firms spread maturity dates over time). First, the model matches many of the stylized facts on average maturities. For example, when period cash flows are riskier, the average repayment time decreases, consistent with classic empirical evidence on debt maturity in Barclay and Smith (1995), Stohs and Mauer (1996), and Custódio, Ferreira, and Laureano (2013). Higher profitability, on the other hand, is associated with more backloaded repayments, consistent with the evidence in Guedes and Opler (1996), Qian and Strahan (2007), and Custódio, Ferreira, and Laureano (2013). Higher leverage is associated with earlier repayment, consistent with evidence on the debt structure of leveraged buyout deals in Axelson, Jenkinson, Strömberg, and Weisbach (2013), as well as the evidence in Billett, King, and Mauer (2007). Second, the model matches the key findings of the recent literature on debt granularity. In particular, consistent with the findings in Choi, Hackbarth, and Zechner (2018, 2017), our model predicts more granular debt structures for mature firms, as well as firms with riskier cash flows, higher leverage, and lower profitability.

In the interest of tractability, our baseline model abstracts away from savings, refinancing, and discounting. However, as we show in a series of extensions, the key results of our model remain robust to these more general settings. Most importantly, the general structure of the optimal contract derived in the baseline model remains pledgeability-maximizing when the firm can save cash flow for later periods or refinance its debt contract.

Our paper contributes to the literature on optimal debt contracts. We build on the literature on debt as a termination threat (in particular, Bolton and Scharfstein (1990, 1996); Hart and Moore (1995, 1998); Berglöf and von Thadden (1994)). While these papers highlight the importance of short-term debt (relative to asset maturity), the two-period nature of these models does not lend itself to the study of the optimal repayment structure when multiple repayment dates are possible. A few papers have extended termination-threat models to more periods. For example, Hart and Moore (1994) characterize the fastest and slowest way to repay in a deterministic multiperiod setting, but because of the absence of default risk, their model does not pin down the number and timing of repayments, which is the focus of our paper. Hart and Moore (1989) offer some examples of optimal debt structure in a three-period model with uncertainty. However, they do not provide a general model of multiperiod debt structure.

Our approach differs from the literature on optimal financial contracting in dynamic settings (e.g., Gromb (1994); DeMarzo and Fishman (2007); Biais, Mariotti, Plantin, and Rochet (2007); DeMarzo and Sannikov (2006)).[2] These papers derive the optimal financing contract in dynamic settings. In contrast, we restrict the contracting space to debt contracts, which allows us to derive

[2]See also the surveys by Biais, Mariotti, and Rochet (2013) and Sannikov (2013).

a rich set of novel predictions on optimal multiperiod debt structure.

More broadly, our paper is also related to the literature on debt maturity, albeit with a different focus. We study how a firm's debt structure (including maturity) emerges from the inability to observe cash flows. In contrast, classic theories of debt maturity have focused on private information (Flannery (1986); Diamond (1991, 1993)), whereas the more recent literature has highlighted strategic interaction among creditors (Cheng and Milbradt (2012)), the inability to commit to financing policies (Brunnermeier and Oehmke (2013); He and Milbradt (2016); DeMarzo and He (2016)), and debt overhang (Diamond and He (2014)).

Finally, our paper is related to a series of papers by Rampini and Viswanathan (2010, 2013), who develop a multiperiod model of financing subject to enforcement constraints. A key difference to our paper is the assumption regarding exclusion. In Rampini and Viswanathan (2010, 2013), no exclusion is possible, such that the optimal contract can be implemented by one-period state-contingent debt contracts. In our paper, liquidation by creditors effectively excludes the entrepreneur from future investment, creating a nontrivial role for debt contracts of different maturities. Moreover, in these models, like in the models of dynamic financing by Albuquerque and Hopenhayn (2004) and Clementi and Hopenhayn (2006), the ability to write fully state-contingent debt contracts implies that there is no default on the equilibrium path. In contrast, the possibility of equilibrium default when debt contracts are not state contingent is a key feature of our model.

1 Model Setup

Consider a risk-neutral entrepreneur who seeks to undertake an investment project. At date $t = 0$, the investment requires an outlay of I. The entrepreneur has cash at hand c and must therefore finance the remainder $D = I - c$ by raising outside financing from competitive, risk-neutral creditors. For simplicity, we assume there is no time discounting.

If funded, the project lasts for T periods. At each date $t \in \mathcal{T} \equiv \{1, 2, \ldots, T\}$, the project generates a random period cash flow X_t. For simplicity, we assume that period cash flows are independent draws from a binary distribution. In particular, with probability $\frac{1}{K}$, the project generates positive cash flow of $K\Delta$, where $\Delta > 0$ and $K \in \mathbb{Z}_+$.[3] With probability $1 - \frac{1}{K}$, the period cash flow at date t is zero. Therefore, the project generates an expected period cash flow of Δ, while the parameter K captures the riskiness of the period cash flow. As we will see, assuming a binary cash-flow distribution with a zero cash flow in the low state makes the analysis particularly tractable and allows us to highlight the key trade-offs in a transparent fashion. However, it is not without loss of generality, and we therefore extend our analysis to more general cash-flow distributions in Section 3.

The main contracting friction in our model is that cash flow is privately observed by the entrepreneur. Therefore, at any date t, the entrepreneur can abscond with the cash flow that was realized in that period, so that payments from the entrepreneur to creditors must be incentive compatible. Incentive compatibility is achieved by a termination threat imposed by creditors via the financing contract. We make three key assumptions about the contracting environment. The first is that we restrict our attention to debt contracts. This distinguishes our analysis from the literature on optimal contracting in dynamic settings (e.g., DeMarzo and Fishman (2007); Biais, Mariotti, Plantin, and Rochet (2007); DeMarzo and Sannikov (2006)). Second, following Hart and Moore (1995), we assume that creditors can commit to liquidating the firm when a debt repayment is missed. As shown by Gromb (1994), the self-defeating nature of renegotiation implies strong ex ante incentives for the firm and creditor to commit to a liquidation strategy.[4] At the same time, the amount of commitment required for our liquidation assumption seems realistic. For example, one simple way to implement liquidation with probability one after a missed payment is to issue widely dispersed debt, like in Diamond and Rajan (2001). Alternatively, a single creditor can commit not to renegotiate by purchasing sufficient credit default swap protection from a third party, like

[3]The assumption that K is an integer is for mathematical convenience. Our results are virtually unchanged without that assumption, except for the added notational complexity of having to deal with integer constraints.

[4]Gromb (1994) shows that, in a multiperiod setting, the ability to repeatedly renegotiate the debt contract severely constrains pledgeability, up to the point where no financing is possible because of anticipated future renegotiation. Because of the creditors' ability to commit to liquidate, this issue does not arise in our framework.

in Bolton and Oehmke (2011).[5] Third, for most of our analysis we assume for simplicity that the entrepreneur can neither carry forward cash balances in the firm nor refinance the debt contract entered at date 0. Therefore, at each date the entrepreneur consumes the period cash flow minus debt repayments made at that date. This assumption allows for a particularly tractable characterization of the optimal debt structure, but it is not crucial for our main results. We relax this assumption in Section 5, where we show that adding savings or refinancing does not affect the main economic insights from our model.

The firm's debt structure is characterized by a sequence of promised repayments $\mathcal{R} = \{R_t\}$, $t \in \{1, 2, \ldots, T\}$.[6] If at any date t the entrepreneur has promised a positive repayment $R_t > 0$ but does not pay, the project is liquidated by the firm's creditors. Neither creditors nor the entrepreneur receives any cash flows in or after liquidation (i.e., the project's liquidation value is normalized to zero, and the entrepreneur cannot undertake another investment after liquidation, either because she is excluded from credit markets or because, without the original creditor, she has lost access to the investment project). As long as the entrepreneur makes contractual debt repayments, the project continues and the entrepreneur consumes $X_t - R_t$.

Denoting by V_t the entrepreneur's payoff at the beginning of date t, a debt contract $\mathcal{R}$ is incentive compatible if and only if $R_t \leq V_{t+1}$ for every $t \geq 1$. The entrepreneur's payoff V_t can be written recursively as

$$V_t = \Delta + \Pr(X_t \geq R_t)\left(-R_t + V_{t+1}\right). \tag{1}$$

This recursive formulation reflects that, at each date t, the entrepreneur generates an expected cash flow of Δ and continues to the next period by making the contractual repayment R_t, whenever

[5]The amount of commitment required for unconditional liquidation in case the firm misses a contractual repayment is significantly lower than committing to the optimal liquidation strategy, which is generally probabilistic and time dependent. In particular, this optimal liquidation strategy cannot be implemented via a dispersed creditor structure and therefore implies a much higher level of commitment on behalf of creditors.

[6]The contract $\mathcal{R}$ can be interpreted in a number of ways. In the most narrow interpretation, $\mathcal{R}$ is the payment schedule of a single debt contract that specifies multiple repayments over time. Interpreted more broadly, $\mathcal{R}$ captures the firm's aggregate debt structure, where individual repayments R_t are potentially separate contracts (i.e., a portfolio of loans or bonds). While from a theoretical perspective these two interpretations are equivalent, the broader interpretation will be useful in linking our model to empirical evidence.

$X_t \geq R_t$. Because the firm cannot save or refinance existing debt, it can only use contemporaneous cash flow to repay debt, so that the firm is liquidated at the first instance when $X_t < R_t$.

Given risk neutrality, the entrepreneur chooses the repayment schedule $\mathcal{R}$ to maximize V_1, the value of equity at the beginning of the project. Formally, the entrepreneur's maximization problem is therefore

$$\begin{aligned} & \max\nolimits_{\mathcal{R}} V_1 \\ \text{s.t.} \quad & R_t \leq V_{t+1} \quad \text{(IC)} \\ & \mathcal{D}(\mathcal{R}) = D \quad \text{(IR)}, \end{aligned} \tag{2}$$

where $\mathcal{D}(\mathcal{R})$ denotes the expected payoff to creditors,

$$\mathcal{D}(\mathcal{R}) = \sum_{t=1}^{T} \prod_{s \leq t} \Pr(X_s \geq R_s) R_t. \tag{3}$$

The two constraints associated with the entrepreneur's maximization problem ensure that promised repayments are incentive compatible (IC) and that risk-neutral creditors break even in expectation (IR). Given that the entrepreneur promises a repayment stream of expected value D and is liquidated the first time that $R_t < X_t$ (i.e., in each period the entrepreneur continues with probability $\Pr(X_s \geq R_s)$), we can write the entrepreneur's value function V_1 explicitly as

$$V_1 = \sum_{i=1}^{T} \prod_{s=1}^{i-1} \Pr(X_s \geq R_s) \Delta - D. \tag{4}$$

In addition to the two constraints above, another quasi-constraint enters the entrepreneur's optimization problem. In particular, it is never in the entrepreneur's interest to offer a debt contract that defaults with probability one in any period.[7] Therefore, any debt contract offered by

[7] Suppose, in contrast, that the optimal contract contains a promised repayment $R_t > K\Delta$ for some t. Then the entrepreneur will default with certainty at date t, even if the positive cash flow $K\Delta$ is realized. Then the creditor's IR constraint (2) holds only if the expected total repayments before date t are equal to D. However, if this is the

the entrepreneur will consist of promised payments R_t that are weakly smaller than the positive cash-flow realization in that period,

$$R_t \leq K\Delta. \tag{5}$$

We will refer to condition (5) as the feasibility condition.

The binary cash-flow distribution combined with the assumption that the entrepreneur cannot save or refinance allows us to simplify the above maximization problem. Specifically, with these assumptions in place, the relevant choice variable for the entrepreneur reduces to whether to promise a positive repayment at any particular date t. To see this, denote the set of dates with positive repayments by $\mathcal{Q} \equiv \{t \in \mathcal{T} | R_t > 0\}$, and let $t_i \in \mathcal{Q}$ be the date of the ith positive repayment. Then, as the following lemma shows, it is the timing and number of repayments that matters for the firm, whereas the exact size of each repayment is usually not uniquely determined.

Lemma 1 *For any two incentive-compatible repayment schedules, $\mathcal{R}$ and $\mathcal{R}'$, if $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}')$ and $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, then the entrepreneur is indifferent between $\mathcal{R}$ and $\mathcal{R}'$.*

Intuitively, Lemma 1 states that if two debt contracts $\mathcal{R}$ and $\mathcal{R}'$ have identical repayment dates and the same expected value to creditors, then they yield the same expected payoff to the entrepreneur.[8]

2 Optimal Debt Structure

In this section we develop our baseline model of optimal debt structure. Section 2.1 presents an intuitive derivation that characterizes the economic trade-off that determines the firm's optimal

case, then the entrepreneur would adjust R_t to 0. This adjustment would not change the creditor's IR constraint (2), but would give the entrepreneur a weakly larger payoff (strictly larger if $t \leq T-1$).

[8]Note that this result relies on the binary cash-flow assumption: the probability of making any positive repayment $R_t \in (0, K\Delta]$ is $\Pr(X = K\Delta) = \frac{1}{K}$ (i.e., the probability of a positive cash-flow realization on that date), regardless of the size of the promised repayment. This is reflected in Equation (4) in that any positive repayment R_t enters the entrepreneur's payoff only through the probability of default at date t. Therefore, only the number and timing of repayments is important, but not the size of each individual repayment. The indeterminacy of individual repayment sizes is a special feature of the binary cash-flow distribution. For example, under a lognormal cash-flow distribution, repayments are uniquely pinned down (see Section 3.3).

debt structure. Section 2.2 then formalizes these insights in a general proposition.

2.1 Optimal debt structure: An intuitive derivation

The main trade-off that determines optimal debt structure is that between early liquidation and sufficient pledgeability. On the one hand, the entrepreneur likes to make debt payments as late as possible. By doing so, the project is less likely to be terminated early on, providing the entrepreneur higher expected cash flows. On the other hand, the entrepreneur faces limits as to how late she can credibly promise to make repayments to creditors. Toward the end of the project, the entrepreneur's continuation value is lower, so she has larger incentives to divert the cash flow and default.

To see how this trade-off shapes the firm's optimal debt structure, it is instructive to start by considering a firm with low outside financing needs D. The nature of the optimal debt structure then emerges as we gradually increase the amount of required outside financing. We develop these results using several cases before moving to a general proposition that fully characterizes the optimal debt repayment structure. The numbering of the cases will become clear as we move from case to case.

Case 1-1: $D \in \left(0, \frac{\Delta}{K}\right]$. We start by assuming that the amount of required outside financing D is weakly less than $\frac{\Delta}{K}$. Because $V_{T+1} = 0$, the entrepreneur clearly cannot credibly promise to make a payment to creditors at date T. Therefore, incentive compatibility requires that $R_T = 0$. However, when $D \leq \frac{\Delta}{K}$, the entrepreneur can raise D by offering a single repayment of $R_{T-1} = KD$ at date $T-1$ (i.e., the set of repayment dates is $\mathcal{Q} = \{T-1\}$). This payment is incentive compatible because the entrepreneur's continuation value at $T-1$ is given by $V_T = \Delta$ (the expected cash flow at date T), which, given $D \in \left(0, \frac{\Delta}{K}\right]$, exceeds the required repayment of KD:

$$R_{T-1} = KD \leq V_T = \Delta. \tag{6}$$

A single repayment of KD at date $T-1$ also satisfies the creditor's IR constraint (2), as $\mathcal{D}(\mathcal{R}) = \frac{1}{K}KD = D$.

From Equation (1), we then see that the entrepreneur's continuation value at the beginning of date $T-1$ is given by

$$V_{T-1} = \Delta + \frac{1}{K}(\Delta - KD),$$

so that the overall payoff to the entrepreneur in this case can be written as

$$V_1 = ... = (T-2)\Delta + V_{T-1} = (T-1)\Delta + \frac{1}{K}(\Delta - KD). \tag{7}$$

Note that even though the entrepreneur can potentially choose to make multiple repayments or offer a single repayment before date $T-1$, neither of these options is optimal. Intuitively, promising multiple risky repayments inefficiently increases default risk. Promising repayment earlier than date $T-1$ unecessarily risks premature termination of the project. Therefore, any alternative schedule with a single repayment of KD at $t' < T-1$, which yields a payoff to the entrepreneur of $t'\Delta + \frac{1}{K}[(T-t')\Delta - KD]$, is dominated by (7).

Case 1-2: $D \in \left(\frac{\Delta}{K}, \frac{2\Delta}{K}\right]$. When the required amount of outside financing D exceeds $\frac{\Delta}{K}$, a single repayment of KD at date $T-1$ is no longer incentive compatible: When $KD > \Delta$, this payment would violate the IC constraint (6). To support a higher repayment, the entrepreneur then optimally moves the single repayment date forward to $T-2$. Because now the final two periods' cash flows are left to the entrepreneur, the entrepreneur's payoff from continuing past date $T-2$ is given by $V_{T-1} = 2\Delta$, which provides the upper bound for the incentive-compatible repayment at $T-2$:

$$R_{T-2} = KD \leq V_{T-1} = 2\Delta.$$

Therefore, when the required amount of financing D lies in the interval $\left(\frac{\Delta}{K}, \frac{2\Delta}{K}\right]$, the entrepreneur can raise the required financing with a single repayment at date $T-2$ (i.e., $\mathcal{Q} = \{T-2\}$) . Because any additional repayment date would create unnecessary default risk, a single payment at date $T-2$ is the optimal way to finance the project.

Case 1-K: $D \in \left(\frac{(K-1)\Delta}{K}, \Delta\right]$. It is easy to see that as the amount of required outside financing

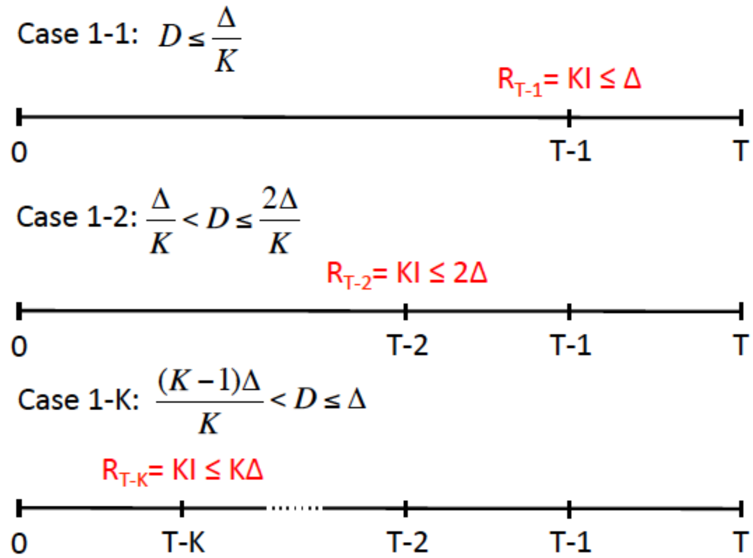

Figure 1: **Financing with one repayment date.** This figure illustrates the range of outside financing needs for which financing is possible with one repayment date, cases 1-1 to 1-K.

D continues to increase, the entrepreneur optimally keeps moving the single repayment forward to maintain incentive compatibility. This is possible as long as the required single repayment satisfies the feasibility condition (5). This leads us to the last case in which financing with one repayment date is possible, case 1-K, with a repayment of KD at date $T-K$ (i.e., $\mathcal{Q} = \{T-K\}$). Analogous to before, the entrepreneur's continuation value $V_{T-K+1} = K\Delta$ allows for a maximum incentive-compatible repayment $R_{T-K} = K\Delta$. However, note that at this point the feasibility constraint (5) also binds, such that moving the repayment forward even further no longer allows the entrepreneur to increase the promised face value. Therefore, $D = \Delta$ is the maximum amount of outside financing that can be raised with a single repayment. Figure 1 summarizes the cases in which financing with only one repayment date is possible (Cases 1-1 to 1-K).

Case 2-1: $D \in \left(\Delta, \Delta + \frac{\Delta}{K^2}\right]$. When the amount of outside financing D exceeds Δ, it is no longer possible to finance the project with a single repayment because the required repayment would violate the feasibility condition (5). Therefore, the entrepreneur must now promise repayments at two dates. The optimal way to do this is to move forward the existing repayment date from $T - K$ to $T - K - 1$ and add a second repayment at date $T - 1$, resulting in optimal repayment dates $\mathcal{Q} = \{T - K - 1, T - 1\}$. Note that once there are two repayment dates, the size of each repayment is no longer uniquely determined, except when D is at the upper bound of the interval (i.e., $D = \Delta + \frac{\Delta}{K^2}$). One possible contract, the slowest way to repay, is to set the final repayment to the maximum incentive-compatible amount, $R_{T-1} = \Delta$, and then set $R_{T-K-1} = KD - \frac{\Delta}{K}$.[9] One can easily verify that this contract satisfies creditor's IR condition (2),

$$\mathcal{D}(\mathcal{R}) = \frac{1}{K} R_{T-K-1} + \frac{1}{K^2} R_{T-1} = D.$$

Moreover, the contract is incentive compatible: For any $D \in \left(\Delta, \Delta + \frac{\Delta}{K^2}\right]$, the IC constraint at date $T - 1$ is clearly satisfied,

$$R_{T-1} = K^2(D - \Delta) \leq \Delta = V_T.$$

To check the IC constraint at date $T - K - 1$, note that, using (1), we can write the continuation value after date $T - K - 1$ as

$$V_{T-K} = \Delta + V_{T-K+1} = ... = (K - 1)\Delta + V_{T-1} = K\Delta + \frac{1}{K}(-R_{T-1} + \Delta).$$

Because R_{T-1} is incentive compatible, we have $V_{T-K} \geq K\Delta \geq R_{T-K-1}$, such that R_{T-K-1} is incentive compatible. Intuitively, leaving K periods of cash flow between the two repayment dates to the entrepreneur ensures that the repayment of $K\Delta$ at date $T - K - 1$ is incentive compatible. The second repayment at date $T - 1$ is bounded by Δ, by exactly the same intuition used in case 1-1. Finally, it is also easy to verify that the schedule $\mathcal{R}$ with $R_{T-K-1} = K\Delta$ and $R_{T-1} = \Delta$

[9] Alternatively, the fastest way to repay is to set the earlier repayment such that it just satisfies the feasibility constraint, $R_{T-K-1} = K\Delta$, and set the second repayment to raise the remainder, $R_{T-1} = K^2(D - \Delta)$. As a result, any equilibrium contract satisfies $R_{T-K-1} \in \left(K\Delta - \frac{\Delta}{K}, K\Delta\right]$, imposing relatively tight bounds on the size of repayments.

attains the upper bound of $D = \Delta + \frac{\Delta}{K^2}$ in this case.

Case 2-2: $D \in \left(\Delta + \frac{\Delta}{K^2}, \Delta + \frac{2\Delta}{K^2}\right]$. As we increase D further, the optimal repayment dates shift forward to $\mathcal{Q} = \{T-K-2, T-2\}$. Specifically, compared with case 2-1, both repayment dates are moved forward by one period. This increases pledgeability at the second (and last) repayment date, while maintaining incentives to repay at the first repayment date. Similar to case 1-2, the maximum incentive-compatible repayment at $T-2$ is $V_{T-1} = 2\Delta$. Keeping K periods between repayments maintains the incentive compatibility of the first repayment. The debt contract with $R_{T-K-2} = K\Delta$ and $R_{T-2} = 2\Delta$ then attains the upper bound of this case, $D = \Delta + \frac{2\Delta}{K^2}$.

Case 2-K: $D \in \left(\Delta + \frac{(K-1)\Delta}{K^2}, \Delta + \frac{\Delta}{K}\right]$. As we continue to increase D, at some point we arrive at Case 2-K, which is the last case in which the required amount of financing can be raised with two repayment dates, which occur at $\mathcal{Q} = \{T-2K, T-K\}$. To raise the maximum amount of outside financing with two repayment dates, the entrepreneur offers repayments of $R_{T-2K} = R_{T-K} = K\Delta$, which attains the maximum debt value of $\Delta + \frac{\Delta}{K}$, the upper bound of case 2-K. At this point, the feasibility condition for both repayments binds. To borrow more, the entrepreneur has to again increase the number of repayment dates. Figure 2 illustrates Cases 2-1 to 2-K.

Based on the pattern that emerges above, we are now in a position to characterize the **general case N-j**, which has N repayments with the final repayment occurring at date $T-j$. Assume that the amount of required outside financing falls into the interval

$$D \in \left(\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}, \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}\right].$$

Then the optimal repayment dates are given by $\mathcal{Q} = \{T-(N-1)K-j, T-(N-2)K-j, ..., T-j\}$ and the maximum feasible and incentive-compatible repayments are $R_{T-nK-j} = K\Delta$ for all $n = 1, 2, ..., N-1$, and $R_{T-j} = j\Delta$ for the final repayment at date $T-j$. The expected value of this debt contract is

$$\mathcal{D}(\mathcal{R}) = \frac{1}{K}K\Delta + \frac{1}{K^2}K\Delta + ... + \frac{1}{K^{N-1}}K\Delta + \frac{1}{K^N}j\Delta,$$

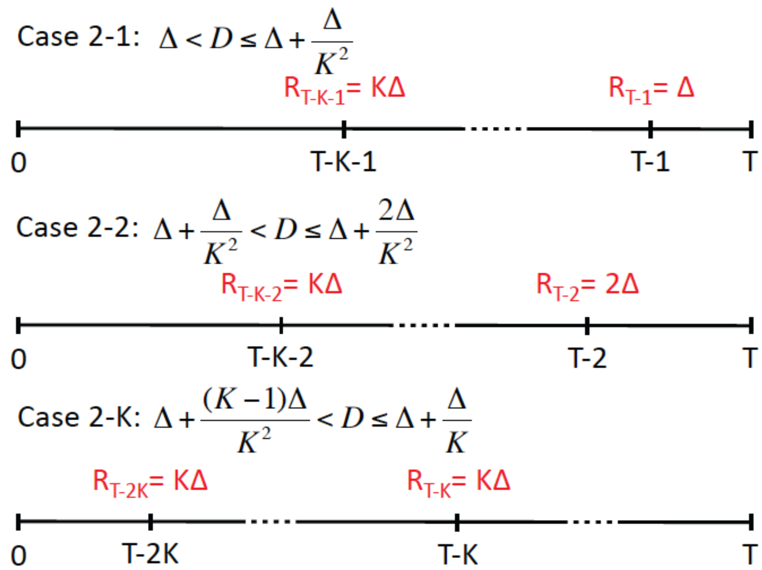

Figure 2: **Financing with two repayment dates.** This figure illustrates the range of outside financing needs for which financing is possible with two repayment dates, Case 2-1 to Case 2-K.

which is equal to the maximum amount of financing that can be raised in Case N-j. Figure 3 illustrates Case N-j.

2.2 Optimal debt structure: General characterization

Based on Case N-j, we can now give a full characterization of the optimal repayment schedule $\mathcal{Q}$.

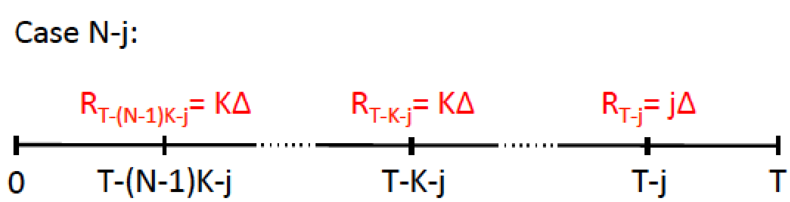

Figure 3: **Financing with N repayment dates.** This figure illustrates the general case with N repayment dates, with the last repayment occurring at date $T - j$, Case N-j.

Proposition 1 *In an optimal debt contract, the set of repayment dates is*

$$\mathcal{Q}_{N,j} \equiv \{T-j, T-K-j, T-2K-j, ..., T-(N-1)K-j\}$$

if and only if the required investment is

$$D \in \left(\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}, \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N} \right], \tag{8}$$

which is a partition of all feasible investment amounts when (N, j) is any pair of positive integers such that

$$T - 1 - (N-1)K - j \geq 0.$$

In addition, when D equals any one of the cutoff values $\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$, the unique optimal debt repayment schedule is given by

$$R_t = \begin{cases} K\Delta, & \text{if } t \in \mathcal{Q} \backslash \{T-j\} \\ j\Delta, & \text{if } t = T-j \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1 characterizes the firm's optimal debt structure. For a given amount of required outside financing D, the proposition uniquely characterizes the optimal repayment dates, as well as the optimal payment amounts at each repayment date at the boundary of each interval in (8). At the boundaries, the incentive-compatibility constraints bind at each of the repayment dates, such that it is impossible to shift repayments between repayment dates in $\mathcal{Q}$. In between the boundaries of the intervals in (8), the repayment dates are still uniquely determined, but the repayment amounts are not uniquely determined. As shown in Lemma 1, the entrepreneur is then indifferent between all incentive-compatible (and feasible) repayment patterns based on the repayment dates $\mathcal{Q}$.

One key feature of the optimal repayment schedule is that, once the firm starts making repayments, these repayments are evenly spaced, separated by K periods to ensure incentive compatibility. Intuitively, because each additional repayment date adds a discrete amount of additional default risk, firms do not smooth their repayments across all periods. Instead, it is optimal to minimize the number of repayments, subject to incentive-compatibility and feasibility constraints.

Denoting by $PI(N)$ the maximum pledgeable income of a debt contract with $N \leq \frac{T-1}{K}$ risky repayment dates, pledgeable income takes the form of a simple geometric sum,

$$PI(N) = \Delta \sum_{n=0}^{N-1} \frac{1}{K^n}. \tag{9}$$

Intuitively, the maximum the firm can pledge with N repayment dates is N repayments of $R_{T-nK} = K\Delta$, each weighted by the probability of making the n^{th} repayment $\frac{1}{K^n}$, where $n = 1, 2, ..., N$. Therefore, pledgeable income is maximized by offering as many risky repayments as possible.

3 More General Cash-Flow Distributions

In this section, we extend the baseline to more general cash-flow distributions. We proceed as follows. First, in Section 3.1, we allow for growth in the project's cash flow. Second, in Section 3.2, we allow for a risk-free cash-flow component. In contrast to the results in Section 2, in both of these cases, pledgeable income is generally no longer maximized by offering as many risky repayments as possible. Rather, pledgeable income is generally largest under a contract that limits the number of risky repayment dates to strictly less than the maximum feasible number. The resultant contract then resembles risky long-term debt, in extreme cases with just one risky bullet repayment. Moreover, firms with a risk-free cash-flow component smooth out at least part of their debt structure by offering safe repayments every period. Finally, in Section 3.3, we discuss continuous cash-flow distributions and provide a numerical solution for the case in which the period cash flow follows a lognormal distribution. Under the lognormal distribution, we show that high cash-flow volatility leads to a lumpy optimal debt structure, where repayments are relatively large and spaced apart. In contrast, when cash-flow volatility is low, the optimal debt structure is smooth, in the sense that it features a debt payment in every period once the firm starts making repayments.

3.1 Cash-flow growth

Suppose that the positive cash-flow realizations grow at the rate $\mu > 1$. Specifically, at any date $t \in \mathcal{T}$, the cash flow is given by $X_t \in \{K\mu^t\Delta, 0\}$. Like in the baseline model, the probability of receiving a positive cash flow $K\mu^t\Delta$ at date t is $\frac{1}{K}$, such that the expected cash flow at date t is $\mu^t\Delta$.

This cash-flow distribution differs from the baseline model mainly in that the maximum feasible repayment now depends on the time when the particular repayment is made. In contrast, in the baseline model, the maximum feasible repayment $K\Delta$ is time invariant. The specification with growth in cash flow may be particularly relevant for young firms, growth firms, and other situations in which the firm's capacity to produce cash flow increases over time.

Similar to the baseline model, where we assumed that K is an integer, we now make an analogous assumption on the pair of (K, μ).

Assumption 1 *There exists $m \in \mathbb{Z}_+$ such that*

$$K = \sum_{s=1}^{m} \mu^s. \tag{10}$$

Assumption 1 ensures that it is incentive compatible for the firm to repay the maximum feasible amount $K\mu^t\Delta$ at t if the next m periods' cash flows are left to the entrepreneur,

$$K\mu^t\Delta = \mu^{t+1}\Delta + \mu^{t+2}\Delta + ... + \mu^{t+m}\Delta.$$

As a result, it is incentive compatible for the entrepreneur to repay $K\mu^t\Delta$ every m periods.

Some of the main insights from the baseline model remain valid with growth in cash flow. As before, the entrepreneur would like to minimize the number of risky repayments and schedule them as late as possible, subject to maintaining incentive compatibility. Moreover, once the firm starts making repayments, these are evenly spaced. The slight difference is that cash-flow growth allows risky repayments to be scheduled closer to each other, at intervals of $m < K$.

Despite these similarities, one key implication of the baseline model changes when we allow for growth in cash flow. Whereas in the baseline model pledgeable income is maximized by scheduling as many repayments as possible (recall Equation (9)), in the presence of cash-flow growth, increasing the number of repayments no longer necessarily increases pledgeable income. To see this, note that the key to increasing pledgeable income by introducing an additional repayment is that the value of existing repayments, which are shifted forward to accommodate the additional repayment, remains unchanged. Figure 4 illustrates that this is no longer the case when there is cash-flow growth. In particular, when cash flows grow over time, positive cash-flow realizations are smaller in earlier periods, such that existing repayments have to be scaled down when they are shifted forward. Therefore, whether adding an additional repayment increases pledgeable income depends on which effect dominates: the decrease in the value of existing repayments that are being shifted forward or the value of the additional repayment that is added at the end.

When the number of existing repayments N is large, the reduction in expected repayments from the first N repayment dates dominates: Shifting forward the existing N repayments by one period reduces their values by the growth factor μ. On the other hand, the value of the additional repayment (e.g., $\mu^T \Delta$ at date $T-1$ in Figure 4) is weighted by the probability that the firm survives past the first N repayments, $\frac{1}{K^N}$, and therefore becomes arbitrarily small when N is large. As a result, for large N a further increase in the number of repayments decreases pledgeable income. Pledgeability is then maximized with N^* repayments, where $N^* > 0$ is given by the smallest integer such that

$$\left[(\mu^{-1} - 1) K \sum_{j=1}^{N^*} \left(K \mu^{-m} \right)^j \right] + 1 < 0. \tag{11}$$

Because $\mu > 1$ and $K\mu^{-m} > 1$ (by the definition of m above), N^* is well defined and unique.

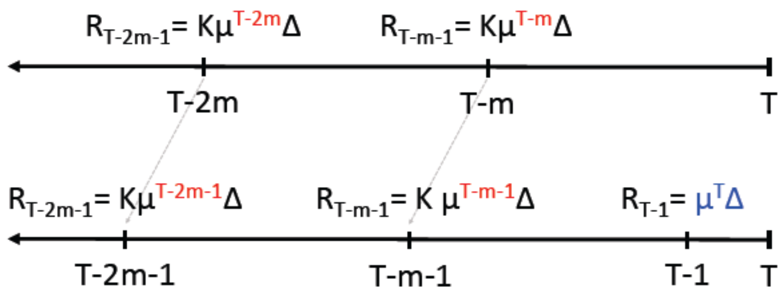

Figure 4: **Cash-flow growth.** In the presence of cash-flow growth, shifting forward existing repayments and adding an additional repayment can reduce pledgeable income. Existing repayments that are shifted forward need to be reduced by the growth factor μ to preserve feasibility. The resultant reduction in the value of all existing repayments outweighs the extra pledgeable income generated by the additional repayment (equal to $\frac{1}{K^N}\mu^T\Delta$) when the number of existing repayments N is sufficiently large.

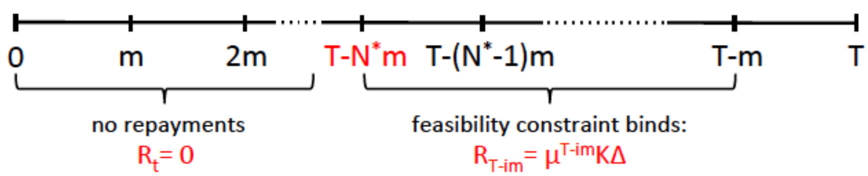

Figure 5: **Pledgeability-maximizing debt structure in the presence of cash-flow growth.** When there is growth in cash flow, pledgeable income is generally maximized with a fixed number of N^* repayments toward the end of the project.

Importantly, N^* is independent of T. Therefore, even when the maturity of the firm's assets and, therefore, the number of possible repayment dates T grows large, pledgeability continues to be maximized with a fixed number of N^* repayments, as illustrated in Figure 5.

Proposition 2 *In the model with growing cash flows ($\mu > 1$),*

1. *the pledgeable income with N repayments $PI(N)$ is maximized at N^* for any T sufficiently large;*

2. *for any $N \leq N^*$,*

$$PI(N) = \sum_{i=0}^{N-1} \frac{\mu^{T-(N-i)m}}{K^i}\Delta;$$

3. *for any* $N \leq N^*$, *if* $D \in (PI(N-1), PI(N)]$, *the optimal debt contract has* N *repayment dates and has the first repayment date* $t_1 \geq T - Nm$;

4. *for any* $N \leq N^*$, *if* $D = PI(N)$, *there is a unique optimal debt contract characterized by*

$$R_t = \begin{cases} \mu^t K \Delta, & \text{if } t \in \{T-m, T-2m, ..., T-Nm\} \\ 0, & \text{otherwise.} \end{cases}$$

A key implication of Proposition 2 is that, for some firms, it may never be optimal to schedule more than one repayment.

Corollary 1 *When* $1 > \frac{1}{K} + \mu^{-m}$, *the maximum number of repayment dates* N^* *is* 1.

From Proposition 2 and Corollary 1, we see that, in the presence of cash-flow growth, the optimal debt contract resembles long-term debt. Independent of the project's horizon T, all repayments occur in the final N^*m periods. In particular, as T becomes large, the earliest possible repayment date under the optimal debt contract, $T - N^*m$, approaches T, in the sense that $\lim_{T\to\infty} \frac{T-N^*m}{T} = 1$.

The finding that long-term debt can maximize pledgeability puts an interesting twist on our understanding of the link between debt maturity and pledgeability. In two-period models, for the threat of termination to be credible, debt has to be short-term, one-period debt. When many repayment dates are possible, on the other hand, it is possible that the debt maturity that maximizes pledgeability roughly matches the project's horizon, especially when T is large. Depending on parameters, our model can therefore capture incentives to finance with short average maturities (leading to maturity mismatch) as well as incentives to finance with longer-term debt (approximate matching of the maturities of assets and liabilities).

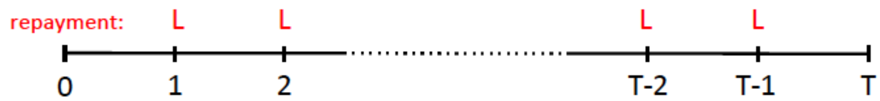

Figure 6: **Risk-free repayment profile in the presence of a significant risk-free cash-flow component.** When $L \geq \frac{\Delta}{K-1}$, pledgeable income is maximized by making a risk-free repayment of L in every period.

3.2 A risk-free cash-flow component

In this section, we extend the model to allow for a risk-free cash-flow component L. Specifically, we assume that the cash-flow distribution X_t is binary with a high cash flow of $L + K\Delta$ with probability $\frac{1}{K}$ and a low cash flow of $L > 0$ with complementary probability. The average per-period cash flow is therefore $\Delta + L$. We assume that L is subject to the same enforceability issues as the risky cash-flow component (i.e., also repayments from the risk-free cash-flow component need to be incentive compatible).

Obviously, if $D \leq (T-1)L$, the optimal debt structure is to repay using risk-free cash flows up to L at every $t \in \{1, 2, ..., T-1\}$. One can also easily verify that any risk-free repayment profile is indeed incentive compatible. More generally, Proposition 3 shows that when the risk-free cash-flow component is sufficiently large, it is never optimal to use risky debt, and pledgeable income is maximized by making a risk-free repayment at every date, as illustrated in Figure 6.

Proposition 3 *If*

$$L \geq \frac{\Delta}{K-1}, \tag{12}$$

then the risk-free schedule $R_t = L$ for all $t \in \{1, 2, ..., T-1\}$ maximizes pledgeable income.

The intuition behind Proposition 3 is as follows. The benefit of increasing repayments beyond the risk-free level is that the entrepreneur pays back more when the high cash flow is realized, which improves pledgeable income. However, this risky repayment also generates default risk, which hurts the expected value of the current as well as all subsequently scheduled repayments. Therefore, risky repayments are never optimal when the loss of a risk-free repayment of L is larger than the expected gain of adding a risky repayment $\frac{\Delta+L}{K}$, as implied by condition (12). Note that condition (12) is more likely to hold when cash-flow risk is large (high K). When default risk is higher, risk-free debt is more likely to be optimal.

For the remainder of this section, we assume (12) does not hold, in order to focus on the case in which introducing risky repayments can increase pledgeability. Like in the case with cash-flow growth analyzed in Section 3.1, here some of the baseline results continue to hold. Specifically, all risky repayments continue to be scheduled toward the end of the project. In addition, to minimize the number of risky repayments, all of them are set to the entire high-cash-flow realization of $K\Delta + L$ and are spaced K periods apart. However, similar to the case with cash-flow growth, we again find that pledgeable income is maximized by limiting the number of risky repayments, in this case to $N^{**} > 0$, where N^{**} is the smallest integer for which the value of a lost risk-free repayment is larger than the value gained from an additional risky repayment,

$$\frac{\Delta + L}{K^{N^{**}+1}} < L,$$

as illustrated in Figure 7.

Proposition 4 *In the presence of a risk-free cash-flow component L,*

1. *the pledgeable income $PI(N)$ is maximized with N^{**} repayment dates for any T sufficiently large;*

2. *for any $N \leq N^{**}$,*
$$PI(N) = (T - 1 - NK)L + \sum_{j=1}^{N} \frac{\Delta + L}{K^{j-1}};$$

3. *for any $N \leq N^{**}$, if $D \in (PI(N-1), PI(N)]$, the optimal debt contract has N repayment dates, and the first risky repayment will be made at date $t_1 \geq T - NK$;*

4. *for any $N \leq N^{**}$, if $D = PI(N)$, there exists a unique optimal contract that is characterized*

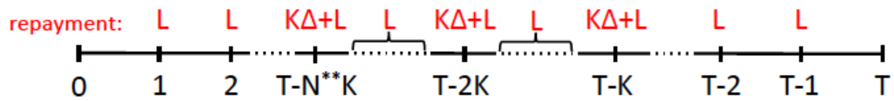

Figure 7: **Risky repayment profile in the presence of a risk-free cash flow component.** When $L < \frac{\Delta}{K-1}$, pledgeability is maximized by limiting the number of risky repayment dates to N^{**}. Risky repayments are scheduled toward the end of the project and spaced K periods apart.

by

$$R_t = \begin{cases} K\Delta + L, & \textit{if } t \in \{T-K, T-2K, ..., T-NK\} \\ L & \textit{otherwise.} \end{cases}$$

Part 3 of Proposition 4 shows that, in the presence of a risk-free cash-flow component, scheduling as many risky repayments as possible does not generally maximize pledgeability. While this result is similar to the case with cash-flow growth, the intuition for limiting the number of risky repayments is slightly different. As the entrepreneur moves the repayment schedule forward by one period to increase risky repayments, she sacrifices one period with a risk-free repayment of L. The contribution to the firm's pledgeable income from the final risky repayment is weighted by the probability of making this repayment $\frac{1}{K^N}$ (if there are N risky repayments). Therefore as N becomes very large, the benefit from the last risky repayment diminishes exponentially, and the cost of sacrificing a risk-free repayment of L dominates.

Part 4 of Proposition 4 shows that a risk-free cash-flow component leads to incentives to partially smooth repayments over time. While the firm continues to offer only periodic risky repayments (if any), the firm pays out the risk-free cash-flow component L in every period.

3.3 Continuous cash-flow distributions

In this section, we relax the assumption of a binary period cash-flow distribution and extend our analysis to continuous distributions. The analysis demonstrates that the firm's optimal debt structure continues to be determined by the economic trade-off highlighted by the baseline model in Section 2 and the extension to a risk-free cash-flow component in Section 3.2. In particular, we show that when the cash-flow distribution is lognormal, high cash-flow volatility leads to a lumpy optimal debt structure, under which repayments are relatively large and spaced apart. In contrast, when cash-flow volatility is low, the optimal debt structure is smooth, in the sense that it features a relatively small debt payment in every period once the firm starts making repayments.

Because of the additional complexity of dealing with continuous distributions in our multiperiod setup, most of the analysis is this section is numerical. However, before proceeding to the numerical analysis, we analytically establish a necessary condition on the distribution from which period cash flows are drawn, such that lumpy repayment profiles can become optimal. Like in the baseline model, we assume that period cash flows are independent and identically distributed (IID).

Define the single-period pledgeability-maximizing face value as

$$\overline{R} \equiv \arg\max_{R} P(X_t \geq R)R.$$

This face value maximizes the amount raised by a single repayment (assuming this payment is incentive compatible). Clearly, the firm has no reason to ever offer a repayment that exceeds $\overline{R}$, because any repayment $R_t > \overline{R}$ raises less financing but results in higher default risk than a repayment of $\overline{R}$. Therefore, if $\overline{R}$ is smaller than the expected cash flow in the next period,

$$\overline{R} \leq E(X_t), \tag{13}$$

then the incentive-compatibility constraint is trivially satisfied in every period, without the need to space out repayments over multiple periods. In this case, the optimal debt structure is smooth: Once repayments start, they occur every period. Moreover, the optimal repayment profile is gradually increasing; earlier repayments are smaller in order to reduce default risk in early periods when defaulting is particularly costly.

If condition (13) is violated, such that $\overline{R} > E(X_t)$, then, like in the baseline model, it can be optimal for the entrepreneur to offer relatively large repayments $R_t \in (E(X_t), \overline{R}]$. These larger repayments must be separated by smaller or zero repayments in order to preserve the borrower's incentive to repay. As the following numerical analysis shows, the lumpy debt structure of the baseline model then emerges when the period cash flow has sufficient upside, whereas smoothing of debt repayments is optimal when the upside in the period cash flow is limited.

For our numerical analysis, we assume that period cash flows follow an IID lognormal distribution.[10] Throughout the analysis, we normalize the mean of the periodic cash flow to $E(X_t) = 1$.[11] For our numerical calculations, we discretize the distribution of the cash flow X_t and the repayments R_t at intervals of 0.1 and then use a numerical algorithm to solve for the optimal debt structure for different amounts of required outside financing in a setting with 11 periods ($T = 11$). Appendix B provides a detailed description of the algorithm.

Figure 8 shows optimal debt structures for different outside financing needs when the period cash flow has relatively high volatility, matching the 90th percentile of asset volatility documented in Frank and Goyal (2009) (corresponding to an asset volatility of 62%).[12] In this case, the period cash-flow distribution has significant upside. Like in the baseline model, the optimal debt structure emerges from the back: when only a small amount of financing has to be raised, the firm offers one repayment at the penultimate date (top-left panel). As the outside financing need increases, more repayment dates are added and are often spaced two periods apart. Pledgeability is maximized with six repayments at dates 1, 3, 5, 7, 9 and 10 (bottom-right panel.) Figure 9, in contrast, shows the optimal debt structure when the period cash flow has relatively low volatility, matching the 90th percentile of asset volatility documented in Frank and Goyal (2009) (corresponding to an asset volatility of 3%).[13] In this case, there is limited upside in the period cash-flow distribution,

[10] The lognormal cash-flow distribution is attractive because it is the workhorse distribution in many structural corporate finance models. It guarantees a strictly positive, unbounded cash flow.

[11] By the properties of the lognormal distribution, this implies that the underlying normal distribution is of the form $N\left(-\frac{\sigma^2}{2}, \sigma\right)$.

[12] Asset volatility is calculated as the return volatility of the period cash flow relative to date-0 unlevered firm value $T{*}E(X)$. Following this calculation, an asset volatility of 62% corresponds to a volatility parameter for the underlying normal distribution of $\sigma = 1.9646$, which gives $\frac{\sqrt{e^{\sigma^2}-1}}{T*E(X)} = \frac{\sqrt{e^{1.9646^2}-1}}{11*1} = 0.62$, using the fact that $\mu = -\frac{\sigma^2}{2}$.

[13] Following the same calculation as before, an asset volatility of 3% corresponds to a volatility parameter for the

but overall the cash flow is relatively safe. Again, the optimal debt structure emerges from the back as the required amount of outside financing increases, but in this case it is smooth: Once repayments start, they occur in every period except the last one (in which no repayment can be extracted). The larger the financing need, the more and larger the repayments that are offered. The pledgeability-maximizing debt structure (bottom-right panel) features a repayment in every period, except at the final date.

Therefore, when the period cash flow is sufficiently volatile, under a lognormal cash-flow distribution the optimal debt structure features spacing that is similar to the baseline model (Proposition 1) and the model with a significant risky cash-flow component (Proposition 4). In contrast, when cash-flow volatility is relatively low, then under the lognormal distribution the optimal debt structure features smoothing, similar to the case with a significant risk-free cash-flow component (Proposition 3), where the upside from offering a higher face value is limited.

4 Empirical Implications

In this section, we discuss the key empirical implications of our model. We first link our results to the empirical literature on debt maturity, which has mainly focused on the average maturities of firms' debt liabilities. We then turn to the more recent literature that has examined the granularity of corporate debt (i.e., how firms spread maturity dates over time). Our model, which provides a unified framework to study debt maturity and debt dispersion from first principles, matches the key empirical findings of these two literatures.

4.1 Average maturity

In our model, the expected average repayment time under the optimal debt contract, taking into account the possibility of default, is given by $\frac{1}{D}\sum_{t=1}^{T} t\frac{R_t}{K^t}$. This measure is similar to classic measures of duration, in that it weights different repayment dates by the present values of the

underlying normal distribution of $\sigma = 0.3216$.

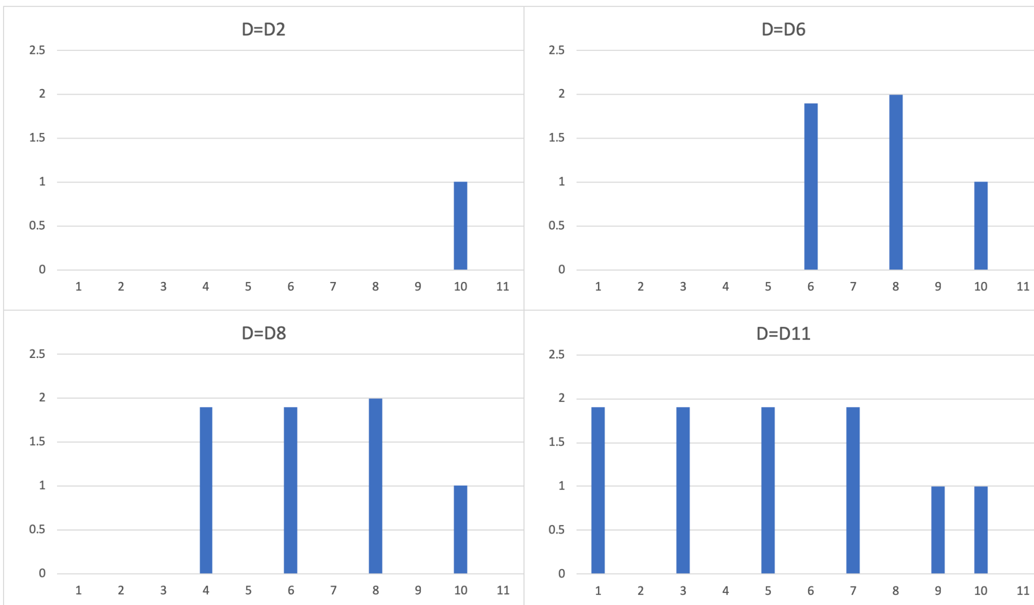

Figure 8: **Lumpy debt structure with lognormal cash flow.** This figure shows the optimal debt structure for different amounts of outside financing when the period cash flow follows a lognormal distribution with 62% asset volatility, corresponding to the 90th percentile in Frank and Goyal (2009). The top-left panel shows the optimal 11-period debt structure when the amount raised is equal to the maximum that could be raised with just two periods ($D = D2$). In the remaining panels, the amount of outside financing is then increased to the maximum that could be raised with 6, 8, and 11 periods (denoted by $D6$, $D8$, and $D11$, respectively). The bottom-right panel therefore corresponds to the pledgeability-maximizing debt structure with 11 cash-flow dates.

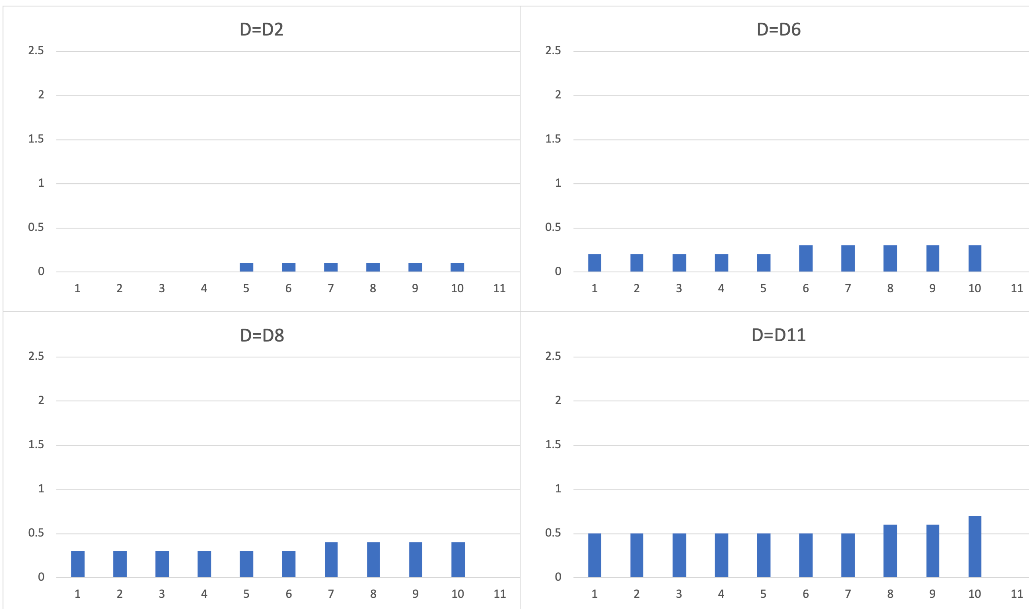

Figure 9: **Smooth debt structure with lognormal cash flow.** This figure shows the optimal debt structure for different amounts of outside financing when the period cash flow follows a lognormal distribution with 3% asset volatility, corresponding to the 10th percentile in Frank and Goyal (2009). The top-left panel shows the optimal 11-period debt structure when the amount raised is equal to the maximum that could be raised with just two periods ($D = D2$). In the remaining panels, the amount of outside financing is then increased to the maximum that could be raised with 6, 8, and 11 periods (denoted by $D6$, $D8$, and $D11$, respectively). The bottom-right panel therefore corresponds to the pledgeability-maximizing debt structure with 11 cash-flow dates.

corresponding repayments. One slight complication is that, as shown in Proposition 1, the exact repayment amounts at each repayment date, and therefore the average repayment time, are not pinned down uniquely except at the point where the debt structure maximizes pledgeable income for a given number of repayment dates.[14] For expositional purposes, we therefore focus on the longest average repayment time, which is achieved by backloading repayments as much as possible. The results for the shortest average repayment time are similar, with very minor modifications.[15]

Lemma 2 *Given any sustainable amount D of outside financing, the longest average repayment time is attained by maximally backloading repayments, $R_{t_i} = K\Delta$ for all $2 \leq i \leq N-1$, $R_{t_N} = j\Delta$, and $R_{t_1} = K(D - \sum_{i=1}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N})$, and can be written as*

$$ART \equiv [T - j - (N-1)K] + \frac{1}{D}\left[\sum_{i=1}^{N-2} \frac{\Delta}{K^{i-1}} i + \frac{j\Delta}{K^{N-1}}(N-1)\right]. \tag{14}$$

In addition to characterizing average repayment times, our model also makes predictions about the number and spacing of repayments. These predictions, which we also point to in the predictions laid out below, are, to the best of our knowledge, largely untested.

4.1.1 Cash-flow risk.

Our first set of empirical predictions relates to the riskiness of cash flows, which in our model is captured by the parameter K. The effect of cash-flow risk on the optimal debt structure is an immediate corollary of Proposition 1: When cash flow becomes riskier, the repayment profile consists of larger promised repayments ($K\Delta$) and longer time intervals (K) between two repayment dates. In addition, maximum pledgeable income with N repayments $PI(N)$ is decreasing in cash-

[14] Repayment dates are uniquely determined. However, except at the boundaries between cases, there are generally multiple schedules of repayment amounts that raise the required amount of outside financing.

[15] As an alternative to our duration measure, one could simply calculate the average scheduled repayment time, not adjusted for the probability of default, $\frac{\sum_{t=1}^{T} R_t t}{\sum_{t=1}^{T} R_t}$. Focusing on the most backloaded contract, the comparative statics under this alternative definition are qualitatively the same as those under the measure that adjusts for default risk. We use the default-risk-adjusted measure, because it is closer to standard measures of duration, which also adjust for default through the yield. Moreover, the proofs are simpler when adjusting for default risk.

flow risk. As a result, to raise the same amount of outside financing, a firm with riskier cash flows needs to spread its debt repayments across more repayment dates. Combined with the longer intervals between repayments, this implies that the entire repayment profile of the optimal debt contract extends forward. Consequently, the average repayment time decreases.

Prediction 1 (Cash-flow risk and average maturity) *As cash-flow risk K increases, holding all other parameters constant, the number of repayments N weakly increases, the time between two repayments increases, and the average repayment time decreases.*

The prediction that cash-flow risk is associated with earlier average repayment has broad support in the empirical literature on debt maturity. Stohs and Mauer (1996) find that riskier firms (as measured by lower EBITDA volatility) have shorter-maturity debt. Barclay and Smith (1995) document that higher volatility of asset returns (implied from equity returns) correlates negatively with the fraction of debt that matures in more than three years. Guedes and Opler (1996) document that in industries with more volatile ROA growth, the maturity of newly issued debt is shorter. Custódio, Ferreira, and Laureano (2013) document that debt maturity correlates negatively with asset volatility.

4.1.2 Profitability.

Next, we examine the effect of profitability on the average repayment time. The expected period cash flow Δ is a natural measure of profitability. However, given the binary cash-flow structure, a change in Δ also affects the variance of the period cash flow, $\Delta^2 (K-1)$. To analyze the marginal effect of higher profitability, we therefore increase Δ while holding cash-flow variance constant by reducing K. This reduction in K implies that the comparative static with respect to profitability is the inverse of Prediction 1: The debt profile of a more profitable company features fewer repayments and shorter intervals between repayment dates, concentrated toward the end of the project's life. The average repayment time is therefore longer.

Prediction 2 (Profitability and average maturity) *As the expected period cash flow* Δ *increases, holding cash-flow variance* $\Delta^2 (K-1)$ *and all other parameters constant, the number of repayments weakly decreases, the time between risky repayments decreases, and the average repayment time increases.*

The prediction that higher profitability is associated with more backloaded repayments is consistent with the evidence that profitability is generally associated with longer debt maturity. For example, Qian and Strahan (2007) find that more profitable firms (as measured by net income divided by assets) borrow longer term. Similarly, Guedes and Opler (1996) show that less profitable firms (as measured by larger operating loss carry-forwards) tend to issue debt of shorter maturity, whereas Custódio, Ferreira, and Laureano (2013) document that abnormal earnings are associated with longer debt maturities.

4.1.3 Leverage.

Finally, we analyze the effect of leverage. The easiest way to analyze higher leverage in our model is through a reduction of the firm's cash resources c. Less cash at hand directly translates into a higher required amount of outside financing $D = I - c$, while leaving the expected cash flows of the firm's project unchanged. From Proposition 1 we know that the entrepreneur can increase the amount of financing raised in three ways: (1) by increasing promised repayments within a given case $N - j$; (2) by moving existing repayment dates forward (an increase in j); and (3) by adding more repayment dates (an increase in N). In the latter two scenarios, the optimal debt contract clearly becomes more short term, in the sense that the average repayment time of the optimal debt contract decreases. However, even in the first case, in which the entrepreneur simply raises face values for a given set of repayment dates (i.e., within a given case $N - j$), the average repayment time decreases. This is because repayments are maximally backloaded, so that higher D increases the first repayment, which reduces the average repayment time.

Prediction 3 (Leverage and average maturity) *Holding all other parameters constant, an increase in leverage (higher D) weakly increases the number of repayments N and decreases the average repayment time.*

Empirically, our model therefore predicts that the debt structures of highly levered firms are more front-loaded, with more repayment dates. As a result, high leverage is predicted to be associated with shorter debt duration. This finding is consistent with the evidence on the debt structure of leveraged buyout deals in Axelson, Jenkinson, Strömberg, and Weisbach (2013), who document that repayment profiles of buyout deals are more front-loaded during times when deals are highly levered. Similarly, Billett, King, and Mauer (2007) document that the proportion of short-term debt increases with firms' leverage.[16]

4.2 Debt granularity

In a series of recent papers, Choi, Hackbarth, and Zechner (2018, 2017) have investigated the granularity of corporate debt. The key object of interest in this literature is the choice between "granular" (or "dispersed") maturity structures (repayments are spread out over time) and "concentrated" maturity structures (with repayments clustered at, say, one or two dates). Our model provides a novel angle to this literature by generating predictions for debt granularity from first principles, based on the same incomplete contracting friction that underpins many of the leading theories of debt financing.

In defining debt granularity, we follow Choi, Hackbarth, and Zechner (2018, 2017), who employ two measures: (1) the inverse of the Herfindahl index of a firm's debt issues, and (2) the squared distance between a firm's maturity profile and a perfectly dispersed maturity profile with equal fractions of face value maturing on each date. Based on both of these measures, our model makes

[16]Interestingly, Barclay and Smith (1995) and Stohs and Mauer (1996) document a positive correlation between leverage and debt maturity. Relative to Axelson, Jenkinson, Strömberg, and Weisbach (2013), the reason for this difference may be that buyouts, where the entire repayment structure is optimized at the time of the deal, correspond more closely to the setting in our paper than studies that investigate the average maturity of a firm's existing debt at a given point in time.

the following predictions.

Prediction 4 (Debt Granularity) *The granularity of the optimal debt contract*

1. *increases with cash-flow risk K, particularly for high-leverage firms;*

2. *is higher, the larger the risk-free cash-flow component L;*

3. *is higher for firms with high leverage D;*

4. *is higher for less profitable firms.*

These predictions are consistent with the evidence in Choi, Hackbarth, and Zechner (2018, 2017). First, Choi, Hackbarth, and Zechner (2018) show that higher rollover risk after the Ford/GM downgrade in 2005 led to more dispersed new debt issuance, particularly for high-leverage firms. Consistent with this finding, in our model higher K, which increases the firm's likelihood of being unable to make a given debt repayment, leads to a debt structure with more repayment dates N and therefore higher debt granularity.[17] Moreover, our model predicts that the increase in debt dispersion is particularly large for highly-levered firms for which the number of repayment dates is particularly sensitive to cash-flow risk K.[18] Moreover, Choi, Hackbarth, and Zechner (2017) document a positive association between profit volatility and debt granularity. They also show that mature firms have more dispersed maturity structures, consistent with the model's prediction that firms spread out the repayments based on the safe cash-flow component L, and firms with higher leverage and lower profitability have more dispersed debt structures, consistent with the final two predictions. Finally, while Choi, Hackbarth, and Zechner (2018, 2017) mostly focus on the granularity of a firm's entire stock of existing debt, Norden, Roosenboom, and Wang (2016) document similar patterns for new debt issuances.

[17]For example, denoting the fraction of face value due at date t by w_t, the squared deviation from the perfectly smooth maturity structure can be written as $D2 = -log \sum_{i=1}^{T}(w_t - 1/T)^2 = -log(1/N - 1/T)$, which is increasing in the number of repayment dates N.

[18]However, note that, in our model, the firm does not in fact reduce rollover risk by offering more repayment dates. Rather, to raise the same amount of outside financing in the presence of higher cash-flow risk, firms have to offer more repayments because each individual repayment is less likely to be made.

5 Extensions

5.1 Savings

In the baseline model, we assumed for simplicity that the entrepreneur can only use contemporaneous cash flow to make repayments. In this section, we show that the optimal debt structure derived in the baseline model (repayments of $K\Delta$ spaced K periods apart once repayments start and, potentially, a smaller final repayment of $j\Delta$ at date $T-j$ with $j \leq K$) remains pledgeability maximizing when the entrepreneur can save. This is a general result, but for expositional purposes we illustrate the intuition through an example rather than providing a full formal treatment.

Like in the baseline model, we assume that the firm raises an amount D to finance the project at date 0. The main difference relative to the baseline model is that, when saving is possible, the entrepreneur can use cash flow from previous periods to make repayments. This has two key implications. First, savings allow the entrepreneur to potentially make payments that exceed the period cash flow (i.e., $R_t > K\Delta$ does not necessarily violate the feasibility condition). However, as we show below, it is never optimal to offer repayments that exceed $K\Delta$. Second, savings introduce a nontrivial trade-off between repayment timing and default risk. Specifically, when the entrepreneur can save, it can be optimal to delay the first repayment in order to allow the firm to accumulate cash. Therefore, relative to the baseline model, the pledgeability-maximizing debt structure may feature fewer (but less risky) repayments. Nevertheless, the general form of the optimal debt structure derived in the baseline model remains pledgeability maximizing when the entrepreneur can save.

We first show that it is never optimal to offer repayments that exceed $K\Delta$, even though the ability to save makes such repayments feasible in principle. For the sake of example, consider a larger repayment of $R_t = 2K\Delta$. Suppose this is the last repayment, such that incentive compatibility requires that this repayment must occur on or before date $T-2K$. Now consider splitting up this single repayment of $2K\Delta$ into two separate repayments of $K\Delta$ at dates $T-2K$ and $T-K$. It is easy to see that this new schedule is also incentive compatible. More importantly, though, splitting the repayment strictly improves the payoff to the entrepreneur. For any realization of cash flows, if the entrepreneur can repay $2K\Delta$ at date $T-2K$, she can also make the two repayments of $K\Delta$ under the new schedule. But the new schedule allows the entrepreneur strictly more time to repay the second $K\Delta$, thereby reducing the probability of liquidation. By the same logic, any debt contract featuring individual repayments that exceed $K\Delta$ can be improved upon.

Proposition 5 *In the optimal debt contract with savings, any individual repayment R_t is weakly smaller than $K\Delta$.*

We now discuss why the general shape of the optimal debt structure derived in the baseline model remains pledgeability maximizing even when the entrepreneur can save. The intuition is similar to that in the baseline model. First, to maximize pledgeability, the incentive constraint must be binding at every repayment date. Second, to minimize default costs, it is optimal to backload repayments as much as possible. Together with the result that even in the presence of savings all repayments are bounded by $K\Delta$, these two conditions lead to a debt structure with repayments of $K\Delta$, every K periods once repayments start, and a final repayment of $j\Delta$ at date $T-j$.[19] Therefore, even though the ability to save significantly reduces default and increases pledgeable income, the pledgeability-maximizing debt structure is analogous to that in the baseline model.

The main difference to the baseline model is that, in the presence of savings, it is no longer necessarily pledgeability maximizing to offer as many repayments as possible. The reason is that it can be optimal to allow the borrower to save cash flow for some periods in order to make future repayments safer and therefore more valuable. Intuitively, relative to the baseline model, moving

[19] Using the recursive formulation of Equation (1) and backward induction, one can verify that the entrepreneur's continuation value on any repayment date t is $V_t = \Delta + Pr(\text{no default})(-R_t + V_{t+1}) = \Delta$ whenever the incentive constraint binds ($R_t = V_{t+1}$). Therefore, on repayment dates, the continuation value V_t is equal to the continuation value in the baseline model, even though the ability to save leads to a significant reduction in the default probability. It then follows from backward induction that also on nonrepayment dates the entrepreneur's continuation value is equal to the continuation value calculated in the baseline model.

repayments forward has an additional cost because it reduces the ability to build up a cash buffer stock. To see the intuition for this result, consider the following example.

Example 1: Suppose that the firm's project lasts for five dates $t = 1, \ldots, 5$, and at each date t, the potential cash flow X_t is either 2Δ or 0 with equal probability (i.e., $K = 2$). In the baseline model without savings, the pledgeability-maximizing contract is $\{2\Delta, 0, 2\Delta, 0, 0\}$. Under the assumptions of the baseline model, the first and second repayments are made with probabilities $\frac{1}{2}$ and $\left(\frac{1}{2}\right)^2$, respectively, so that the date-0 value of this debt contract is

$$\frac{1}{2} * 2\Delta + \left(\frac{1}{2}\right)^2 * 2\Delta = \frac{3}{2}\Delta.$$

When the entrepreneur can save, the probability of making the second repayment increases. Conditional on making the first repayment, the second repayment of 2Δ is made if X_2 or X_3 (or both) is positive. Hence, with savings the value of the contract $\{2\Delta, 0, 2\Delta, 0, 0\}$ increases to

$$\frac{1}{2} * \left\{2\Delta + \left[1 - \left(\frac{1}{2}\right)^2\right] * 2\Delta\right\} = \frac{7}{4}\Delta.$$

However, the entrepreneur can increase pledgeability even further by postponing the first repayment and reducing the second repayment to preserve incentive compatibility: $\{0, 2\Delta, 0, \Delta, 0\}$, which, as one can easily check, is the pledgeability-maximizing contract. Under this contract, the first repayment of 2Δ is made with probability $\left[1 - \left(\frac{1}{2}\right)^2\right] = \frac{3}{4}$. The second repayment of Δ is made with certainty if $X_1 = X_2 = 2\Delta$ (with probability $\frac{1}{4}$) . If only one of X_1 and X_2 is positive (with probability $\frac{1}{2}$), making the second repayment requires a positive cash flow in X_3 or X_4, which happens with probability $\frac{3}{4}$. The value of this debt contract is therefore

$$\frac{3}{4} * 2\Delta + \frac{1}{4} * \Delta + \frac{1}{2} * \frac{3}{4} * \Delta = \frac{17}{8}\Delta > \frac{7}{4}\Delta.$$

Hence, while the structure of the pledgeability-maximizing debt contract is the same as in the baseline model, it is shifted backward so that the firm can build up a cash buffer.

Finally, note that, even though the optimal debt structure of the baseline model remains pledgeability maximizing, the ability to save cash flow generally allows for multiple ways to implement a given pledgeable income. In particular, in the presence of a savings technology one can usually split the repayments of $K\Delta$ into smaller repayments that are spread out over K periods. In the previous example, the repayment profiles $\{0, 2\Delta, 0, \Delta, 0\}$ and $\{0, \Delta, \Delta, \Delta, 0\}$ are equivalent because, if the entrepreneur can repay 2Δ at date 2 in the first contract, she can make two payments of Δ at dates 2 and 3 in the second contract. Conversely, if the entrepreneur can make the first repayment of Δ in the second contract, she has received at least one cash-flow realization of 2Δ, so she could also make a repayment of 2Δ that would be required under the first contract. However, a small repayment cost or an imperfection in the savings technology would restore unique optimality of the debt structure derived in the baseline model.

5.2 Refinancing

In this section, we present an extended version of the baseline model in which the entrepreneur can refinance the outstanding debt at any point before the end of the project's life. The main result of this section is that, without loss of generality, we can focus on contracts under which the firm is liquidated the first time it receives a zero cash flow on a scheduled repayment date (i.e., like in the baseline model).

Unlike the ability to save, which allows the entrepreneur to use cash flow from previous periods, refinancing enables the entrepreneur to use (unpledged) future cash flow to make a repayment today. To capture this feature, denote by $\mathcal{R}^s = \{(R_t^s, B_t^s) | t > s\}$ the debt contract signed at date s, which specifies a schedule of repayments R_t^s to be paid at dates $t > s$ and a schedule of lump-sum refinancing payments B_t^s that must be paid to creditors in case the debt contract is terminated early (i.e., B_t^s is the call price of all debt outstanding at date t). We use the convention $B_t^s = \infty$ to indicate instances when the contract signed at date s does not allow for refinancing at date t.

Using the notation introduced above, the initial contract signed at date 0, $\mathcal{R}^0$, is a schedule of repayment amounts and call prices $\{(R_t^0, B_t^0) | t > 0\}$. Subsequently, at each date t, the cash flow $X_t \in \{0, K\Delta\}$ is realized. If the borrower repays R_t^0, the contract continues on to date $t+1$. Alternatively, the borrower can choose to terminate the current contract by paying B_t^0. If the cash flow X_t alone is insufficient to pay B_t^0, the borrower can refinance the remainder $B_t^0 - X_t$ by issuing a new contract $\mathcal{R}^t$. Denoting by $\mathcal{D}(\mathcal{R}^t)$ the value of the refinancing contract $\mathcal{R}^t$, taking into account all possible refinancing in the future, successful refinancing therefore requires that $\mathcal{D}(\mathcal{R}^t) = B_t^0 - X_t$. We assume that, at date 0, the entrepreneur can commit to a (grand) refinancing plan $\{\mathcal{R}^s | s \geq 1\}$, with each refinancing contract $\mathcal{R}^s$ depending on the history of publicly observable variables (i.e., previous contracts and refinancing decisions).[20]

Relative to the baseline model, one possibility that arises in the presence of refinancing is that the firm may be able to survive a period with a positive scheduled repayment $R_t^s > 0$ and a zero-cash-flow realization $X_t = 0$ by refinancing to a new contract $\mathcal{R}^t$ that satisfies $\mathcal{D}(\mathcal{R}^t) = B_t^s$. We will refer to such a repayment as a *refinanceable repayment*, because a zero cash-flow realization on the repayment date no longer leads to termination. The main result of this section is that, without loss of generality, we can focus on contracts without such refinanceable repayments. Under a contract without refinanceable repayments, the firm is liquidated the first time it has a zero cash-flow realization on a repayment date, as is the case in the baseline model.

The intuition behind this result is that, for any refinancing plan with refinanceable repayments, we can construct an equivalent plan without refinanceable repayments. The key step in the construction is to eliminate the refinanceable repayment and replace the original continuation contract following the refinanceable repayment by the refinancing contract starting from that date. To illustrate this in the simplest possible way, we return to the five-period example from Section 5.1. However, the reasoning behind the example applies more generally.

[20]The assumption that a grand refinancing contract is signed at date 0 is not crucial. However, it greatly simplifies the analysis because it means that we do not have to deal with beliefs about future refinancing contracts and possible deviations.

Example 2: Suppose that the entrepreneur's initial borrowing is $D = \Delta$. Consider first an initial contract and associated refinancing plan $\{\mathcal{R}\}$ with an initial repayment profile $R^0 = \{0, \Delta, 0, 0, 0\}$ and $B^0 = \{\infty, \Delta, \infty, \infty, \infty\}$. The first repayment $R_2^0 = \Delta$ is a refinanceable repayment because, even if $X_2 = 0$, the entrepreneur can repay Δ by refinancing to a new contract $\mathcal{R}^2$ with $R^2 = \{2\Delta, 0, 0\}$ and $B^2 = \{\infty, \infty, \infty\}$. Note that, under this new contract, the entrepreneur cannot refinance again if $X_3 = 0$. As a result, the value of the refinancing contract entered at date 2, $\mathcal{R}^2$, is $\frac{1}{2} * 2\Delta = \Delta$, which is sufficient to raise $B_2^0 = \Delta$.

Next, we construct an equivalent refinancing plan $\{\widetilde{\mathcal{R}}\}$ without the refinanceable repayment R_2^0. The key insight is that the refinancing contract $\mathcal{R}^2$ now replaces the initial contract starting from date 3, that is, $\widetilde{R}^0 = \{0, 0, R^2\} = \{0, 0, 2\Delta, 0, 0\}$ and $\tilde{B}_t^0 = B_t^2$ for $t \geq 3$. Intuitively, if a payment is refinanceable in the original contract, we simply use the refinancing contract as part of the original contract. In addition, the refinanceable repayment ($R_2^0 = \Delta$) is replaced by a zero promised repayment and an option to prepay $R_2^0 = \Delta$, that is, $\tilde{B}^0 \equiv \{B_1^0, R_2^0, B^2\} = \{\infty, \Delta, \infty, \infty, \infty\}$. The option to refinance at date 2 is taken only when cash flow $X_2 = 2\Delta$.[21] Under this refinancing plan, if $X_2 = 2\Delta$, the entrepreneur will refinance and thereby effectively prepay $\widetilde{B}_2^0 = \Delta$ to finish required repayments. Note that this is the same effective repayment pattern as under $\{\mathcal{R}\}$, where, if $X_2 = 2\Delta$, the entrepreneur repays $R_2^0 = \Delta$. On the other hand, if $X_2 = 0$, under $\{\mathcal{R}\}$ the entrepreneur refinances, whereas under $\{\widetilde{\mathcal{R}}\}$ the entrepreneur simply continues the ongoing contract and makes a payment of $\tilde{R}_3^0 = 2\Delta$ at date 3 if $X_3 = 2\Delta$ (and defaults if $X_3 = 0$). The effective repayment streams are exactly the same, so that $\{\widetilde{\mathcal{R}}\}$, which does not include any refinanceable repayments, is equivalent to $\{\mathcal{R}\}$, the contract with refinanceable repayments.

Under the plan without refinanceable repayments, $\{\widetilde{\mathcal{R}}\}$, the key additional incentive-compatibility constraint is the manager's willingness to refinance at date 2 if a positive cash flow $X_2 = 2\Delta$ is real-

[21] In general, for a refinanceable repayment R_t^0, the new refinancing payment $\tilde{B}_t^0$ should be the original repayment plus the continuation value of the original contract starting from date $t+1$, that is, $\tilde{B}_t^0 = R_t^0 + D_{t+1}(\mathcal{R}^0)$. The contract $\tilde{\mathcal{R}}^t$ to refinance the difference $D_{t+1}(\mathcal{R}^0) = \tilde{B}_t^0 - R_t^0$ is simply the original contract $\mathcal{R}^0$ starting from date $t+1$. Note that, in the example, the continuation contract $\mathcal{R}^0$ starting from date 3 is $\{0, 0, 0\}$, so the entrepreneur can pay $\tilde{B}_2^0$ using cash flow $X_2 = 2\Delta$ without issuing a refinancing contract $\tilde{R}^2$.

ized. If the manager chooses not to refinance and consume X_2, his payoff is $X_2+E(X_3-\widetilde{R}_3^0+\widetilde{V}_4^0) = 2\Delta + \frac{1}{2}(2\Delta - 2\Delta + 2\Delta) = 3\Delta$. This is strictly less than the payoff under refinancing, which is given by $X_2 - \tilde{B}_2^0 + E(X_3) + E(X_4) + E(X_5) = 2\Delta - \Delta + 3\Delta = 4\Delta$. Hence, if $X_2 = 2\Delta$ the manager will refinance at date 2, as specified in the equilibrium strategy.

5.3 Equivalence between refinancing and savings

Focusing on the debt contract without refinanceable repayments is also helpful in establishing a close parallel between refinancing and savings. Under this contract, refinancing is a way for the entrepreneur to prepay in high-cash-flow states that do not feature a contractual repayment. Given this, it becomes intuitive that the same outcome can be achieved by saving early cash flows (or paying them into an escrow account or sinking fund). In fact, as we illustrate below, the maximum pledgeable income is the same under savings and refinancing. This result implies that the structure of the optimal contract in the baseline model (repayments of $K\Delta$ spaced K periods apart and, potentially, a smaller final repayment of $j\Delta$ at date $T-j$ with $j \leq K$) remains pledgeability maximizing even when refinancing is possible.

To show this, we return to the five-period example from Section 5.1 and characterize the pledgeability-maximizing repayment profile under refinancing.

Example 3: Consider an initial contract $\mathcal{R}^0$ that consists of the promised repayment scheme $R^0 = \{0, 2\Delta, 0, \Delta, 0\}$ and the early termination amounts $B^0 = \{\frac{23}{8}\Delta, \infty, \Delta, \infty, \infty\}$. (Note that refinancing is impossible at dates 2, 4, and 5 because $B = \infty$. Therefore, if the entrepreneur misses the repayment at those dates, the project will be terminated.) At date 1, if $X_1 = 0$, the entrepreneur will continue $\mathcal{R}^0$, whereas if $X_1 = 2\Delta$, the entrepreneur will terminate $\mathcal{R}^0$ by prepaying 2Δ and refinancing $\frac{7}{8}\Delta$ by issuing a new contract $\mathcal{R}^1$ with $R^1 = \{0, 0, \Delta, 0\}$ and $B^1 = \{\Delta, \Delta, \infty, \infty\}$. Intuitively, $\mathcal{R}^1$ requires the entrepreneur to make a payment of Δ at date 4, but allows for prepayment of this amount at dates 2 and 3.

We are now in a position to calculate the value of the above refinancing plan. Suppose first that $X_1 = 0$. If it is also the case that $X_2 = 0$, then the project is terminated and the creditor receives 0. If $X_2 = 2\Delta$, then the entrepreneur pays $R_2^0 = 2\Delta$ and has two more chances to pay an additional Δ, either by prepaying $B_3^0 = \Delta$ or repaying $R_4^0 = \Delta$. Conditional on $X_1 = 0$, the expected payoff to the creditor is therefore

$$\frac{1}{2} * 0 + \frac{1}{2} * \left[2\Delta + \left(1 - \left(\frac{1}{2} \right)^2 \right) \Delta \right] = \frac{11}{8}\Delta.$$

Now suppose that $X_1 = 2\Delta$. In this case, the entrepreneur will repay 2Δ using the current cash flow and refinance $\frac{23}{8}\Delta - 2\Delta = \frac{7}{8}\Delta$ by issuing $\mathcal{R}^1$. $\mathcal{R}^1$ raises $\frac{7}{8}\Delta$ by giving the entrepreneur three chances to repay the remaining face value Δ, either by prepaying B_2^1 or B_3^1 or by making the contractual repayment R_4^1, which is worth $\left(1 - \left(\frac{1}{2}\right)^3\right)\Delta = \frac{7}{8}\Delta$ and therefore raises exactly the amount required for refinancing. Conditional on $X_1 = 2\Delta$, the expected payoff to the creditor (including the value of the repayment and the refinancing contract) is therefore $2\Delta + \frac{7}{8}\Delta = \frac{23}{8}\Delta$. Seen from date 0, the expected value of the initial contract $\mathcal{R}^0$ is then given by

$$\frac{1}{2} * \frac{23}{8}\Delta + \frac{1}{2} * \frac{11}{8}\Delta = \frac{17}{8}\Delta.$$

This (pledgeability-maximizing) refinancing contract therefore achieves the same pledgeable income as the pledgeability-maximizing contract under savings.[22] Moreover, note that the structure of the individual repayment profiles R_0 and R_1 mirrors the optimal repayment profile derived in the baseline model (periodic repayments of $K\Delta$ spaced K periods apart and, potentially, a smaller final repayment of $j\Delta$ at date $T - j$ with $j \leq K$).[23] Of course, because the ability to refinance reduces the incidence of default on the equilibrium path, generally fewer repayments are required to finance the same amount D than under the optimal contract in the baseline model.

[22]Incentive compatibility of the pledgeability-maximizing refinancing contract can be checked in similar fashion to Example 2 above. We omit the details for brevity.

[23]The equivalence between refinancing and savings in the case of the pledeability-maximizing contract is general. We conjecture that the same equivalence holds for values of debt below the pledgeability-maximizing level, but we have not formally established this result.

5.4 Discounting

In this section, we briefly discuss how the introduction of time discounting affects our results. We first discuss introducing a common discount rate and then turn to the case in which the entrepreneur discounts the future more than do creditors.

Common discount rate: The simplest (perhaps obvious) case is when the entrepreneur and creditors have a common discount factor $\beta < 1$ and cash flows grow at the common discount rate (i.e., the high-cash-flow realization on date t is given by $\beta^{-t}K\Delta$). In this case, all results carry over analogously to the baseline model. This is because this case boils down to a simple rescaling of variables, where the rescaled variables are defined as $\widetilde{R}_t = \beta^{-t}R_t$ and $\widetilde{X}_t = \beta^{-t}X_t$. Scheduled repayments will be increasing over time, given that the feasibility constraint is now $R_t \leq \beta^{-t}K\Delta$. Repayments continue to be spaced K periods apart, like in the model without discounting. To see this, note that if we leave K periods until next payment, then the entrepreneur receives

$$\beta\beta^{-t-1}\Delta + \beta^2\beta^{-t-2}\Delta + \cdots + \beta^K\beta^{-t-K}\Delta = \beta^{-t}K\Delta,$$

which makes the highest feasible date-t repayment $R_t = \beta^{-t}K\Delta$ just incentive compatible.

Things are more interesting when there is a common discount rate but cash flows are constant (i.e., the high-cash-flow realization on date t is given by $K\Delta$). In this case, the results of our model continue to hold with some slight adjustments as long as the future is not discounted too heavily, i.e., $\beta \in [\overline{\beta}, 1]$. Intuitively, this case is essentially the opposite of the model with cash-flow growth analyzed in Section 3.1. Like in the case of cash-flow growth, in this case the entrepreneur's IC constraint is affected. In particular, to support a repayment of $K\Delta$ today, the required number of periods until the next repayment is now given by $m > K$ (recall that with cash-flow growth we have the opposite result, $m < K$). Setting aside integer constraints, the distance between repayments m is implicitly pinned down by the condition that today's repayment of $K\Delta$ has to be equal to the expected cash flows that accrue to the entrepreneur over the next m periods,

$$K\Delta = \sum_{i=1}^{m} \beta^i \Delta.$$

Note that by setting $\beta = 1$, we recover the baseline model without discounting ($m = K$).

In addition to changing the spacing between repayment dates to $m > K$, the discounting of promised future repayments affects the creditor's breakeven constraint. All other things equal, in the presence of discounting, each promised future repayment is worth less than in the absence of discounting. Together with the result that repayments are spaced $m > K$ periods apart, discounting therefore implies that more and earlier repayments are required to raise a given amount D relative to the case without discounting.

When the future is discounted heavily, $\beta \in (0, \overline{\beta})$, additional complications can arise. For brevity, we only sketch them out here. First, it may be possible that the feasibility constraint is never binding because the largest incentive-compatible repayment is strictly lower than $K\Delta$. This is the case when β satisfies $K\Delta > \sum_{i=1}^{\infty} \beta^i \Delta = \frac{\beta}{1-\beta}\Delta$, such that even promising all future cash flows to the entrepreneur is not sufficient to make a repayment of $K\Delta$ incentive compatible. Second, with heavy discounting, it is possible that one early repayment of $K\Delta$ can raise the same amount (or more) than two later repayments. If this is the case, then, unlike in the model without discounting, it may no longer be optimal to backload the debt structure as much as possible. For example, with significant discounting it is possible that a debt contract with repayments of $K\Delta$ on dates 1 and 4 raises the same amount of funding as a debt contract with repayments of $K\Delta$ on dates 2, 5, and 8. There is a nontrivial trade-off between the two contracts. Under the first contract, default risk is incurred earlier but, because there are only two repayments, the overall probability of default is lower. Of course, this complication arises only when time discounting is significant. Without discounting (and, by continuity, with moderate discounting), moving repayments forward does not reduce the number of required repayments, such that it remains optimal to backload the debt structure, like in the baseline model. Therefore, as long as $\beta \geq \overline{\beta}$, repayments are backloaded, subject to incentive compatibility, even in the presence of discounting.

Entrepreneur has a higher discount rate than creditors: Finally, suppose that the entrepreneur discounts the future more than creditors do. For simplicity, we focus on the case in which the entrepreneur's discount factor is $\beta^E < 1$ and the creditors' discount factor is $\beta^L = 1$. Consider, as before, an entrepreneur who is raising D to finance an investment. Because the entrepreneur discounts the future more so than do creditors, a second reason to schedule repayments as late as possible arises (in addition to avoiding early liquidation): Relative to the creditors, the entrepreneur values early consumption more highly. However, the ability to schedule repayments as late as possible is limited by the fact that discounting changes the entrepreneur's IC constraint (as discussed above), therefore requiring a spacing of $m > K$ between repayments. Hence, despite the additional desire for late repayments that results from the difference in discounting, the entrepreneur ends up having to make repayments earlier to guarantee incentive compatibility of the promised repayments.

6 Conclusion

This paper provides a model of optimal debt structure in a multiperiod setting. Building on the insights of the literature on debt as a termination threat, which has mostly focused on two-date settings, our multiperiod model shows how a rich optimal debt structure emerges from a simple trade-off between providing the firm with incentives to repay and preventing costly early liquidation.

Our model predicts that firms backload their debt structure subject to incentive compatibility—as they increase their borrowing, they add periodic risky repayments from the back of the maturity structure, with the time between repayments increasing in cash-flow risk. Cash-flow growth or a significant risk-free cash-flow component limits the number of periodic risky repayments because they introduce additional costs of defaulting early. On the other hand, firms with low cash-flow risk choose smooth maturity profiles with relatively safe repayments every period, rather than larger but risky periodic repayments. The model provides a unified framework to study debt maturity and debt granularity, and its main results are robust to a number of extensions, including the introduction of savings, refinancing, or discounting.

References

Albuquerque, R., and H. A. Hopenhayn, 2004, "Optimal lending contracts and firm dynamics," *Review of Economic Studies*, 71(2), 285–315.

Axelson, U., T. Jenkinson, P. Strömberg, and M. S. Weisbach, 2013, "Borrow cheap, buy high? The determinants of leverage and pricing in buyouts," *Journal of Finance*, 68(6), 2223–67.

Barclay, M. J., and C. W. Smith, 1995, "The maturity structure of corporate debt," *Journal of Finance*, 50(2), 609–31.

Berglöf, E., and E.-L. von Thadden, 1994, "Short-term versus long-term interests: Capital structure with multiple investors," *Quarterly Journal of Economics*, 109(4), 1055–84.

Biais, B., T. Mariotti, G. Plantin, and J.-C. Rochet, 2007, "Dynamic security design: Convergence to continuous time and asset pricing implications," *Review of Economic Studies*, 74(2), 345–90.

Biais, B., T. Mariotti, and J.-C. Rochet, 2013, "Dynamic financial contracting," *Advances in Economics and Econometrics; Tenth World Congress of the Econometric Society*, pp. 125–71.

Billett, M. T., T. King, and D. C. Mauer, 2007, "Growth opportunities and the choice of leverage, debt maturity, and covenants," *Journal of Finance*, 62(2), 697–730.

Bolton, P., and M. Oehmke, 2011, "Credit default swaps and the empty creditor problem," *Review of Financial Studies*, 24(8), 2617–55.

Bolton, P., and D. S. Scharfstein, 1990, "A theory of predation based on agency problems in financial contracting," *American Economic Review*, 80(1), 93–106.

——— , 1996, "Optimal debt structure and the number of creditors," *Journal of Political Economy*, 104(1), 1–25.

Brunnermeier, M. K., and M. Oehmke, 2013, "The maturity rat race," *Journal of Finance*, 68(2), 483–521.

Cheng, I.-H., and K. Milbradt, 2012, "The hazards of debt: Rollover freezes, incentives, and bailouts," *Review of Financial Studies*, 25(4), 1070–110.

Choi, J., D. Hackbarth, and J. Zechner, 2017, "Granularity of corporate debt," Working Paper.

——— , 2018, "Corporate debt maturity profiles," *Journal of Financial Economics*, 130, 484–502.

Clementi, G. L., and H. A. Hopenhayn, 2006, "A theory of financing constraints and firm dynamics," *Quarterly Journal of Economics*, 121(1), 229–65.

Custódio, C., M. A. Ferreira, and L. Laureano, 2013, "Why are US firms using more short-term debt?," *Journal of Financial Economics*, 108(1), 182–212.

Dang, T. V., G. Gorton, and B. Holmström, 2015, "Ignorance, debt and financial crises," Working Paper, Columbia, Yale, and MIT.

DeMarzo, P., and Z. He, 2016, "Leverage dynamics without commitment," Working Paper.

DeMarzo, P. M., and M. J. Fishman, 2007, "Optimal long-term financial contracting," *Review of Financial Studies*, 20(6), 2079–128.

DeMarzo, P. M., and Y. Sannikov, 2006, "Optimal security design and dynamic capital structure in a continuous-time agency model," *Journal of Finance*, 61(6), 2681–724.

Diamond, D. W., 1991, "Debt maturity structure and liquidity risk," *Quarterly Journal of Economics*, 106(3), 709–37.

——— , 1993, "Seniority and maturity of debt contracts," *Journal of Financial Economics*, 33(3), 341–68.

Diamond, D. W., and Z. He, 2014, "A theory of debt maturity: The long and short of debt overhang," *Journal of Finance*, 69(2), 719–62.

Diamond, D. W., and R. G. Rajan, 2001, "Liquidity risk, liquidity creation, and financial fragility: A theory of banking," *Journal of Political Economy*, 109(2), 287–327.

Flannery, M. J., 1986, "Asymmetric information and risky debt maturity choice," *Journal of Finance*, 41(1), 19–37.

Frank, M. Z., and V. K. Goyal, 2009, "Capital structure decisions: Which factors are reliably important?," *Financial Management*, 38(1), 1–37.

Gale, D., and M. Hellwig, 1985, "Incentive-compatible debt contracts: The one-period problem," *Review of Economic Studies*, 52(4), 647–63.

Gorton, G., and G. Pennacchi, 1990, "Financial intermediaries and liquidity creation," *Journal of Finance*, 45(1), 49–71.

Gromb, D., 1994, "Renegotiation in debt contracts," Working Paper, INSEAD.

Guedes, J., and T. Opler, 1996, "The determinants of the maturity of corporate debt issues," *Journal of Finance*, 51(5), 1809–33.

Hart, O., and J. Moore, 1989, "Default and renegotiation: A dynamic model of debt," Working Paper Version.

——— , 1994, "A theory of debt based on the inalienability of human capital," *Quarterly Journal of Economics*, 109(4), 841–79.

——— , 1995, "Debt and seniority: An analysis of the role of hard claims in constraining management," *American Economic Review*, 85(3), 567–85.

——— , 1998, "Default and Renegotiation: A Dynamic Model of Debt," *Quarterly Journal of Economics*, 113(1), 1–41.

He, Z., and K. Milbradt, 2016, "Dynamic debt maturity," *Review of Financial Studies*, 29(10), 2677–736.

Innes, R. D., 1990, "Limited liability and incentive contracting with ex-ante action choices," *Journal of Economic Theory*, 52(1), 45–67.

Norden, L., P. Roosenboom, and T. Wang, 2016, "The effects of corporate bond granularity," *Journal of Banking & Finance*, 63, 25–34.

Qian, J., and P. E. Strahan, 2007, "How laws and institutions shape financial contracts: The case of bank loans," *Journal of Finance*, 62(6), 2803–34.

Rampini, A. A., and S. Viswanathan, 2010, "Collateral, risk management, and the distribution of debt capacity," *Journal of Finance*, 65(6), 2293–322.

Rampini, A. A., and S. Viswanathan, 2013, "Collateral and capital structure," *Journal of Financial Economics*, 109(2), 466–92.

Sannikov, Y., 2013, "Dynamic security design and corporate financing," *Handbook of the Economics of Finance*, 2, 71–122.

Stohs, M. H., and D. C. Mauer, 1996, "The determinants of corporate debt maturity structure," *Journal of Business*, pp. 279–312.

Townsend, R. M., 1979, "Optimal contracts and competitive markets with costly state verification," *Journal of Economic Theory*, 21(2), 265–93.

Appendix A. Proofs

Proof of Lemma 1: Because $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}')$, and X_t is either $K\Delta$ or 0, Equation (5) implies that $\Pr(X_t \geq R_t) = \Pr(X_t \geq R'_t)$ for any t. Therefore,

$$\sum_{t=0}^{T}\prod_{s=0}^{t-1}\Pr\left(X_s \geq R_s\right)\Delta = \sum_{t=0}^{T}\prod_{s=0}^{t-1}\Pr\left(X_s \geq R'_s\right)\Delta.$$

In addition, because $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, it follows from Equation (4) that $\mathcal{R}$ and $\mathcal{R}'$ will lead to the same V_1. Therefore, the entrepreneur is indifferent between $\mathcal{R}$ and $\mathcal{R}'$.

Proof of Proposition 1: We prove this proposition by a series of claims.

Claim 1: For any two incentive-compatible debt contracts, $\mathcal{R}$ and $\mathcal{R}'$, if $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$ and $\mathcal{Q} \subset \mathcal{Q}'$, then the entrepreneur strictly prefers $\mathcal{R}$. Put differently, other things equal, the entrepreneur wants to reduce the number of repayments.

To see this, first note that because $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, it follows from Lemma 1 that $V_1(\mathcal{R}) > V_1(\mathcal{R}')$ if and only if

$$\sum_{t=0}^{T}\prod_{s=0}^{t-1}\Pr\left(X_s \geq R_s\right)\Delta - \sum_{t=0}^{T}\prod_{s=0}^{t-1}\Pr\left(X_s \geq R'_s\right)\Delta > 0.$$

Because $\mathcal{Q} \subset \mathcal{Q}'$, $\Pr\left(X_s \geq R_s\right) \geq \Pr\left(X_s \geq R'_s\right)$ for all $s \in \mathcal{T}$. However, because at least one element in $\mathcal{Q}'$ does not belong to $\mathcal{Q}$, there is at least one $s' \in \mathcal{T}$ such that $R_{s'} = 0$ and $R'_{s'} \in (0, K\Delta]$. Hence, at s', $\Pr\left(X_{s'} \geq R_{s'}\right) = 1 > 1/K = \Pr\left(X_{s'} \geq R'_{s'}\right)$. Therefore, the entrepreneur strictly prefers $\mathcal{R}$.

Claim 2: Denote by $\#\mathcal{Q}(\mathcal{R})$ the number of repayments of the debt contract $\mathcal{R}$ and by ϱ the vector of the repayment dates. If $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, $\#\mathcal{Q}(\mathcal{R}) = \#\mathcal{Q}(\mathcal{R}')$, and $\varrho > \varrho'$ (that is, any element of ϱ is greater than or equal to ϱ', and at least one element of ϱ is strictly greater than the corresponding element of ϱ'), then the entrepreneur strictly prefers $\mathcal{R}$. Put differently, if two incentive-compatible debt contracts have the same value and the same number of repayments, the entrepreneur prefers the one with late repayments.

Because $\varrho > \varrho'$, for any t, $\prod_{s=0}^{t-1}\Pr\left(X_s \geq R_s\right) \geq \prod_{s=0}^{t-1}\Pr\left(X_s \geq R'_s\right)$, and there exists a repayment date $t_j \in \mathcal{Q}(\mathcal{R})$ that comes strictly later than the corresponding repayment date $t'_j \in \mathcal{Q}(\mathcal{R}')$. Then, at t_j, $\prod_{s=0}^{t_j-1}\Pr\left(X_s \geq R_s\right) > \prod_{s=0}^{t_j-1}\Pr\left(X_s \geq R'_s\right)$. Therefore, the entrepreneur strictly prefers $\mathcal{R}$.

Claim 3: We next show that the optimal debt contract has exactly N repayments if

$$D \in \left(\Delta\sum_{i=0}^{N-2}\frac{1}{K^i}, \Delta\sum_{i=0}^{N-1}\frac{1}{K^i}\right]. \tag{A1}$$

Before proving claim, we state a repeatedly used adjustment procedure to the debt contract as a lemma.

Lemma 3 *Suppose $t_i, t_j \in \mathcal{Q}(\mathcal{R})$ are two repayment dates, with $R_{t_j} < K\Delta$. Define "(t_i, t_j, ϵ) adjustment" to be the following procedure to construct a new contract $\mathcal{R}'$: $R'_{t_i} = R_{t_i} - \epsilon$ and $R'_{t_j} = R_{t_j} + \frac{\epsilon}{K^{i-j}}$, leaving all other repayments unchanged. Then the value of debt is unchanged, $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. In addition, if $t_i > t_j$, then $\mathcal{R}'$ is also incentive compatible.*

Proof of Lemma 3: First, it is straightforward from (3) that $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. Next, if $t_i > t_j$, let V' be the entrepreneur's continuation value under contract $\mathcal{R}'$. It follows from Equation (1) that the entrepreneur's payoff V_t can be rewritten as

$$V_t = \sum_{i=t}^{T} \prod_{s=t}^{i-1} \Pr(X_s \geq R_s)\,\Delta - \sum_{i=t}^{T} \prod_{s=t}^{i} \Pr(X_s \geq R_s)\,R_i.$$

Then it is clear that $V_t' \geq V_t$ holds for all t and strictly for $t_j < t \leq t_i$. In particular, $V'_{t_j+1} = V_{t_j+1} + \frac{\epsilon}{K^{i-j}}$, so condition $R'_{t_j} \leq V'_{t_j+1}$ still holds. IC conditions for other repayments are trivially satisfied. This completes the proof of Lemma 3.

We now prove Claim 3. Consider any debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) \leq N-1$ first. Let the i^{th} repayment date be $t_i \in \mathcal{Q}$. Note that the maximum amount of any single repayment is $K\Delta$; and R_{t_i} is actually paid if and only if $X_{t_\tau} = K\Delta$ for all $\tau \leq i$, which happens with probability $1/K^i$. Therefore, the maximum total expected repayments of $\mathcal{R}$ with at most $N-1$ repayments is

$$\sum_{i=1}^{N-1} \left[\frac{1}{K^i}(K\Delta)\right] = \Delta \sum_{j=0}^{N-2} \frac{1}{K^i} < D$$

by Equation (A1). Hence, creditor's IR constraint (2) implies that there must be at least N repayments: $\#\mathcal{Q}(\mathcal{R}) \geq N$.

Next, we show that any contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = N+k$, where $k \geq 1$, can be strictly improved. Suppose there is a positive integer $j < N+k$ such that $R_{t_j} < K\Delta$. Then we can apply the $(N+k, j, \epsilon)$ adjustment until either all initial $N+k-1$ repayments equal $K\Delta$ or the last repayment becomes zero, $R'_{t_{N+k}} = 0$. In the first case,

$$\sum_{j=1}^{N+k} \left[\frac{1}{K^j} R_{t_j}\right] > \sum_{j=1}^{N+k-1} \left[\frac{1}{K^j}(K\Delta)\right] \geq D,$$

so the total expected value of repayments exceeds D, contradicting the IR constraint (2). In the second case, the entrepreneur can eliminate the last repayment without affecting the value of debt. By Claim 1, the adjusted repayment schedule is strictly preferred. Hence, we establish Claim 3.

Now, we prove Proposition 1. If D satisfies (8), then it automatically satisfies (A1). Claim 3 implies that the optimal debt contract will include exactly N repayments. Denote the optimal debt contract by $\mathcal{R}^*$.

Next, we inductively prove that the i^{th} repayment occurs at $t_i^* = T - j - (N-i)K$. We first establish the statement for $i = N$, namely, $t_N^* = T - j$. Consider any debt contract $\mathcal{R}$ whose last repayment date is later than $T - j$. The incentive-compatibility constraint implies that $R_{t_N} \leq (j-1)\Delta$. However, because the expected value of the first $N-1$ repayments is at most $\sum_{i=0}^{N-2} \frac{\Delta}{K^i}$ (attained when every repayment is $K\Delta$), $D \leq \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}$, violating Equation (8).

Consider any debt contract $\mathcal{R}$ with the last repayment date $t_N < T - j$. We can apply the (t_N, t_i, ϵ) adjustment until $R_{t_i} = K\Delta$ for all $i < N$. After such an adjustment, condition (8) implies $R_{t_N} \leq j\Delta$. So, the entrepreneur can delay R_{t_N} to $T - j$ without affecting incentive compatibility and the value of debt, which, by Claim 2, makes the entrepreneur strictly better off. Hence, in $\mathcal{R}^*$, $t_N^* = T - j$.

Suppose $t_s^* = T - j - (N-s)K$ for all $s \geq i+1$. We now prove the statement for t_i^*. First, starting from $\mathcal{R}^*$, we can apply the (t_i^*, t_l^*, ϵ) adjustment for all $l < i$, until $R_{t_l} = K\Delta$. Next, apply the (t_i^*, t_l^*, ϵ) adjustment for all $l > i$, until $R_{t_l^*} = K\Delta$ (if $l < N$) or $j\Delta$ (if $l = N$). By Lemma 3, both adjustments do not affect the value of debt, and the first is incentive compatible. It is easy to see that the second adjustment is also incentive compatible because the induction assumption implies that $V_{t_l^*} = K\Delta$ (or $j\Delta$) for all $i < l < N$

(or $l = N$).

After the adjustment,

$$R_{t_i^*} = \left(D - \sum_{l=0, l \neq i-1}^{N-2} \frac{\Delta}{K^l} - \frac{j\Delta}{K^N} \right) K^i.$$

From (8), $R_{t_i^*} \in \left(K\Delta - \frac{\Delta}{K^{N-i}}, K\Delta\right] \subset ((K-1)\Delta, K\Delta]$. Therefore, IC condition for $R_{t_i^*}$ implies that $t_i^* \leq t_{i+1}^* - K$. If $t_i^* < t_{i+1}^* - K$, we can simply move $R_{t_i^*}$ to a later date: $t_{i+1}^* - K$. It is easy to see that such an adjustment does not affect the value of debt and is still incentive compatible. By Claim 2, the new contract dominates $\mathcal{R}^*$, contradicting the optimality of $\mathcal{R}^*$. Therefore, the induction conclusion holds for i and $t_n^* = T - j - (N-n)K$ for any $n \leq N$ in $\mathcal{R}^*$.

Finally, when $D = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$, we know from the previous proof that $\mathcal{Q}(\mathcal{R}^*) = \{T-j, T-j-K, \ldots, T-j-(N-1)K\}$. IC conditions imply that $R_{t_i}^* \leq K\Delta$ for $i < N$ and $R_{t_N}^* \leq j\Delta$. As a result, $\mathcal{D}(\mathcal{R}^*) \leq D$, with equality holding if and only if $R_{t_i}^* = K\Delta$ for $i < N$ and $R_{t_N}^* = j\Delta$. This establishes the uniqueness and completes the proof.

Proof of Proposition 2:

Part 1: Consider first a debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = 1$. We claim that the repayment date $t = T - m$ maximizes $PI(1)$. To see this, we note that R_t has two upper bounds. First, $R_t \leq K\mu^t\Delta$, due to the feasibility constraint; and second, $R_t \leq V_{t+1}$ for the contract to be incentive compatible. Note that

$$V_{t+1} = \sum_{s=t+1}^{T} \mu^s \Delta = \mu^t \sum_{s=1}^{T-t} \mu^s \Delta.$$

Hence, $\forall t \in [T-m, T)$,

$$V_{t+1} \leq \left(\sum_{s=1}^{m} \mu^s \right) \mu^t \Delta = K\mu^t\Delta,$$

and so the incentive-compatibility constraint must be binding to attain the maximum pledgeable income; that is, $R_t = V_{t+1}$. Because V_{t+1} is strictly decreasing in t, when $t \in [T-m, T)$, to achieve the maximum pledgeability, the repayment date must be $T - m$, and the maximum pledgeable income is $K\mu^{T-m}\Delta$.

Now, consider $t \leq T - m$. Then

$$V_{t+1} \geq \left(\sum_{s=1}^{m} \mu^s \right) \mu^t \Delta \geq K\mu^t\Delta.$$

So the feasibility constraint must be binding to maximize $PI(1)$. Because $K\mu^t\Delta$ is strictly increasing in t, to achieve the maximum pledgeability, the repayment date must be $T - m$, and the maximum pledgeable income is $K\mu^{T-m}\Delta$. Combining both cases, if $\#\mathcal{Q}(\mathcal{R}) = 1$, $PI(1) = K\mu^{T-m}\Delta$, which is attained by making the only repayment at date $T - m$. By the same arguments, we show that for a debt contract $\mathcal{R}$, if $t_j \in \mathcal{Q}$ and $R_{t_j} = V_{t_j+1}$ (i.e., $V_{t_j} = \mu^{t_j}\Delta$), then the maximum repayment at the $(j-1)^{\text{th}}$ repayment date occurs at $t_j - m$.

Now, let's consider $PI(2)$. For any debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = 2$, suppose $t_2 = T - q$. It follows from the proof of the one repayment contract that $q \leq m$. Then, to attain the maximum pledgeable income, the entrepreneur can first set

$$R_{t_2} = V_{t_2+1} = \sum_{s=T-q+1}^{T} \mu^s \Delta.$$

As a consequence, $t_1 = t_2 - m$. Denote by $PI^q(N)$ the maximum pledgeable income of a contract with N repayments and the last repayment occurs at date $T - q$. Then

$$PI^q(2) = \frac{K\mu^{T-q-m}\Delta}{K} + \frac{1}{K^2}\sum_{s=T-q+1}^{T}\mu^s\Delta.$$

So,

$$PI^{q+1}(2) - PI^q(2) > 0 \Leftrightarrow \mu^{-1} + \frac{\mu^m}{K^2} > 1. \tag{A2}$$

Suppose Equation (A2) holds, then $PI^q(2)$ is strictly increasing in q. Hence,

$$PI(2) = PI^m(2) = \mu^{T-2m}\Delta + \frac{\mu^{T-m}\Delta}{K}.$$

Let's now compare $PI(2)$ and $PI(1)$ under Equation (A2):

$$PI(2) > PI(1) \Leftrightarrow \mu^{-m} + \frac{1}{K} > 1. \tag{A3}$$

Note that, by Equation (10), we have

$$K - \frac{K}{\mu} = \sum_{s=1}^{m}\mu^s - \sum_{s=0}^{m-1}\mu^s = \mu^m - 1. \tag{A4}$$

Then Equation (A3) is equivalent to $1 - \mu^m + \frac{\mu^m}{K} > 0$, which holds if and only if $\frac{K}{\mu} - K + \frac{\mu^m}{K} > 0$. The last inequality is equivalent to Equation (A2). So, when $n = 2$, $PI(2) = PI^m(2)$ if and only if $PI(2) > PI(1)$.

We now use induction. Assume that $PI(n) = PI^m(n)$ if and only if $PI(n) > PI(n-1)$, where $n \geq 2$. Let's consider $n + 1$. Fix any $t_{n+1} = T - q$. When the contract can attain the largest pledgeable income, $R_{t_{n+1}} = V_{T-q+1}$. Then, by the assumption that $PI(n) = PI^m(n)$, we have

$$PI^q(n+1) = \left[\sum_{j=1}^{n}\frac{\mu^{T-q-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}}\sum_{j=T-q+1}^{T}\mu^j\Delta\right].$$

Then

$$\begin{aligned}
& PI^{q+1}(n+1) - PI^q(n+1) \\
= & \left[\sum_{j=1}^{n}\frac{\mu^{T-(q+1)-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}}\sum_{j=T-(q+1)+1}^{T}\mu^j\Delta\right] \\
& \qquad - \left[\sum_{j=1}^{n}\frac{\mu^{T-q-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}}\sum_{j=T-q+1}^{T}\mu^j\Delta\right] \\
= & \mu^{T-q}\left[(\mu^{-1}-1)\sum_{j=1}^{n}\frac{\mu^{-(n-j+1)m}\Delta}{K^{j-1}}\right] + \frac{\mu^{T-q}\Delta}{K^{n+1}}.
\end{aligned}$$

So, $PI^{q+1}(n+1) > PI^q(n+1)$ if and only if

$$\left[(\mu^{-1}-1)\sum_{j=1}^{n}\frac{\mu^{-(n-j+1)m}}{K^{j-1}}\right]+\frac{1}{K^{n+1}}>0. \tag{A5}$$

Now suppose Equation (A5) holds; then $PI(n+1) = PI^m(n+1)$. We then have

$$PI(n+1)-PI(n)=\sum_{j=1}^{n+1}\frac{\mu^{T-(n-j+2)m}}{K^{j-1}}-\sum_{j=1}^{n}\frac{\mu^{T-(n-j+1)m}}{K^{j-1}}>0$$

if and only if

$$(1-\mu^m)\sum_{j=1}^{n}\frac{\mu^{-(n-j+1)m}}{K^{j-1}}+\frac{1}{K^n}>0. \tag{A6}$$

It then follows from equation (A4) that

$$1-\mu^m=\frac{K}{\mu}-K.$$

Hence, Equations (A6) and (A5) are equivalent. Therefore, if $PI(n+1) = PI^m(n+1)$, $PI(n+1) > PI(n)$.

Note that Equation (A5) is equivalent to

$$\left[(\mu^{-1}-1)K\sum_{j=1}^{n}\left(K\mu^{-m}\right)^{n-j+1}\right]+1>0.$$

It then follows from Equation (11) that if and only if $N \leq N^*$, $PI(N) \geq PI(N-1)$. Therefore, $PI(N)$ is maximized at $N = N^*$.

Part 2: Because for any $N \leq N^*$, $PI(N) = PI^m(N)$. Therefore,

$$PI(N)=\sum_{i=0}^{N-1}\frac{\mu^{T-(N-i)m}}{K^i}\Delta.$$

Part 3: Now, suppose $D \in (PI(N-1), PI(N)]$. By the definition of $PI(N)$, because $D > PI(N-1)$, it is impossible to design a contract with at most $N-1$ repayment dates such that the creditor's IR constraint holds. But $D \leq PI(N)$, so there exists a contract with N repayments such that the creditor's participation constraint holds.

Consider any contract $\mathcal{R}$ with $\#\mathcal{Q} = N + p$ ($p \in \mathbb{Z}_+$). Without loss of generality, we only consider contracts with $R_t = K\mu^t\Delta$, $\forall t \in \mathcal{Q} \setminus \{t_{N+p}\}$. Otherwise, if $R_{t_j} < K\mu^{t_j}\Delta$, the entrepreneur can apply the (t_{N+p}, t_j, ϵ) adjustment until either $R_{t_j} = K\mu^{t_j}\Delta$ or $R_{t_{N+p}} = 0$. The former case is under consideration, while in the latter case, the entrepreneur is strictly better off. Note, in this process, the contract's incentive compatibility and the creditor's participation constraint are preserved.

We now first claim that for any $t_j, t_{j+1} \in \mathcal{Q} \setminus \{t_1\}$, $t_{j+1} - t_j \geq m$. Let $t_j \in \mathcal{Q}$ be the last repayment date at which $t_{j+1} - t_j < m$. Because the original debt contract $\mathcal{R}$ is incentive compatible, it follows from the definition of m in Equation (10) and $R_s = K\mu^s\Delta$ for all $s \in \mathcal{Q} \setminus \{t_{N+p}\}$ that

$$\begin{aligned} & R_{t_j} \leq V_{t_j+1}, \\ \Leftrightarrow & \sum_{s=t_j+1}^{t_j+m} \mu^s\Delta \leq \sum_{s=t_j+1}^{t_{j+1}} \mu^s\Delta + \frac{1}{K^{N+p-j}} \left(\sum_{s=t_{N+p}+1}^{T} \mu^s\Delta - R_{t_{N+p}} \right) \\ \Leftrightarrow & R_{t_{N+p}} \leq \sum_{s=t_{N+p}+1}^{T} \mu^s\Delta - K^{N+p-j} \sum_{s=t_{j+1}+1}^{t_{j+1}+\ell} \mu^s\Delta, \end{aligned}$$

where $\ell = t_j + m - t_{j+1} \geq 1$. If $t_{N+p} + 1 = T$,

$$\begin{aligned} & R_{t_{N+p}} \\ \leq & \sum_{s=t_{N+p}+1}^{T} \mu^s\Delta - K^{N+p-j} \sum_{s=t_{j+1}+1}^{t_{j+1}+\ell} \mu^s\Delta \\ = & \mu^{t_{j+1}} \left[\mu^{T-t_{j+1}}\Delta - K^{N+p-j} \sum_{s=1}^{\ell} \mu^s\Delta \right] \\ < & \Delta\mu^{t_{j+1}} \left[\mu^{(N+p-(j+1))m+1} - \mu^{(N+p-j)m+1} \right] < 0, \end{aligned}$$

where the first strict inequality is again due to the definition of m in Equation (10). Such an inequality contradicts the assumption that $R_{t_{N+p}} > 0$ in the original contract, and so $t_{N+p} + 1 < T$.

Then the entrepreneur can construct a new contract $\mathcal{R}'$ such that $R'_{t_{N+p}} = 0$, $R'_{t_{N+p}+1} = R_{t_{N+p}}$, and $R'_s = R_s$ for all other s. That is, the last repayment is delayed to date $t_{N+p} + 1$. Let's check the incentive compatibility of the new contract $\mathcal{R}'$. Under $\mathcal{R}'$, at date $t_{N+p} + 1$, we have

$$\begin{aligned} & V'_{t_{N+p}+2} - R'_{t_{N+p}+1} \\ = & V'_{t_{N+p}+2} - R_{t_{N+p}} \\ \geq & \sum_{s=t_{N+p}+2}^{T} \mu^s\Delta - \left(\sum_{s=t_{N+p}+1}^{T} \mu^s\Delta - K^{N+p-j} \sum_{s=t_{j+1}+1}^{t_{j+1}+\ell} \mu^s\Delta \right) \\ > & \left(K^{N+p-j} - \mu^{(N+p-j-1)m} \right) \mu^{t_{j+1}+1}\Delta > 0. \end{aligned}$$

Hence, the incentive-compatibility constraint holds at date $t_{N+p}+1$. Because $t_{s+1} - t_s = m$ for all $s \geq j+1$, the incentive-compatibility constraint holds at all these dates. At t_j, because $V'_{t_j+1} > V_{t_j+1}$ (because all subsequent repayment amounts are unchanged, and the last repayment is delayed), $V'_{t_j} > V_{t_j}$. Therefore, the incentive compatibility of $\mathcal{R}$ implies the incentive compatibility of $\mathcal{R}'$. Importantly, the creditor's participation constraint does not change. Therefore, $\mathcal{R}'$ satisfies both the incentive-compatibility constraint and the creditor participation constraint. Finally, because $V'_{t_j} > V_{t_j}$, and all repayments before t_j are the same in $\mathcal{R}$ and $\mathcal{R}'$, $\mathcal{R}$' is strictly better than $\mathcal{R}$. This implies that our original assumption $t_{j+1} - t_j < m$ is invalid.

We then only need to consider a contract with $t_{j+1} - t_j \geq m$ for any $t_j, t_{j+1} \in \mathcal{Q} \setminus \{t_1\}$. Because $t_{N+p} < T$ and $p \geq 1$, $t_{N+p} - (N+p-j)m < T - (N-j+1)m$. Hence, if the entrepreneur offers a contract $\mathcal{R}'$ with $\#\mathcal{Q}' = N$, the j^{th} repayment day could be $T - (N-j+1)m$, which is later than the j^{th} repayment date in the original contract $\mathcal{R}$. Therefore, there exists $\mathcal{R}'$ with $\#\mathcal{Q}' = N$ that is strictly better than $\mathcal{R}$. Therefore, when $D \in (PI(N-1), PI(N)]$, the optimal contract has exactly N repayment dates. In addition, because

$t_{j+1} - t_j = m$ and $T - t_N \le m$, the first repayment date $t_1 \ge T - Nm$.

Part 4: When $D = PI(N)$, the optimal debt contract must have $t_N = T - m$, because $PI(N) = PI^m(N)$. Then $\mathcal{Q} = \{T - m, T - 2m, ..., T - Nm\}$. In addition, to attain the maximum pledgeable income, the entrepreneur must make the largest repayment at each repayment date, so $R_t = K\mu^t\Delta$, implying that the schedule proposed is the unique one to attain $D = PI(N)$, which can be attained by the repayment schedule

$$R_t = \begin{cases} \mu^t K\Delta, & \text{if } t \in \{T - m, T - 2m, ..., T - Nm\} \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Corollary 1: Consider the left-hand side of Equation (11). When $N = 1$, we have

$$\left[(\mu^{-1} - 1)K \sum_{j=1}^{N} \left(K\mu^{-m}\right)^j\right] + 1 = (\mu^{-1} - 1)K(K\mu^{-m}) + 1. \tag{A7}$$

Note that when $(\mu^{-1} - 1)K = 1 - \mu^m$, Equation (A7) becomes $K\mu^{-m} - K + 1$, which is negative because $1 > \frac{1}{K} + \mu^{-m}$. So when $1 > \frac{1}{K} + \mu^{-m}$, $N^* = 1$.

Proof of Proposition 3: Suppose, on the contrary, that a risky contract $\mathcal{R}$ maximizes the value of debt. Denote by $t \le T - 1$ the last risky repayment date. Construct a new contract $\mathcal{R}'$ such that $R'_s = R_s$ for all $s < t$ or $s > t + 1$, $R'_t = L$, and $R'_{t+1} = R_{t+1} + \max(0, R_t - \Delta - L) < R_t$.

The new contract $\mathcal{R}'$ is incentive compatible because, for any s,

$$V'_s \ge (\Delta + L) + \frac{1}{K}(V'_{s+1} - R_s) \ge \Delta + L,$$

so all risk-free repayments $R'_s \le L$ when $s > t+1$ are automatically incentive compatible. In addition, R'_{t+1} is incentive compatible because either $R'_{t+1} = R_{t+1}$, in which case IC trivially holds, or $R'_{t+1} = R_{t+1} + R_t - \Delta - L$ and

$$V'_{t+2} = V_{t+2} = V_{t+1} + R_{t+1} - (\Delta + L) \ge R_t + R_{t+1} - (\Delta + L) = R'_{t+1}.$$

Finally, note that

$$\begin{aligned} & V'_t - V_t \\ \ge & \left[2(\Delta + L) - L + \frac{V'_{t+2} - R'_{t+1}}{K}\right] - \left[(\Delta + L) + \frac{V_{t+2} + (\Delta + L) - R_{t+1} - R_t}{K}\right] \\ = & \Delta + \frac{R_{t+1} + R_t - (\Delta + L) - R'_{t+1}}{K} > 0. \end{aligned}$$

Therefore, we can recursively show that $V'_s > V_s$ for all $s \le t$, and $R'_s = R_s$ is thereby incentive compatible. Next we show $\mathcal{D}(\mathcal{R}') > \mathcal{D}(\mathcal{R})$. This is because, when (12) holds,

$$\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \ge L + \frac{R_{t+2} + \max(0, R_t - \Delta - L)}{K} - \frac{R_{t+1} + R_{t+2}}{K} \ge L - \frac{\Delta + L}{K} > 0.$$

Contradiction with the maximality of $\mathcal{R}$. Therefore, risk-free schedule $R_s = L$ maximizes pledgeability.

Proof of Proposition 4: We prove this proposition in four parts.

Part 1: Let N be the number of risky repayments. We show that the value of any repayment profile with $N \neq N^{**}$ can be strictly improved. Let's first consider a contract $\mathcal{R}$ with $N > N^{**}$. Suppose $t_N < T$ is the last risky repayment date. If there is a $t < t_1 \in \mathcal{Q}$, such that $R_t < L$, the entrepreneur can simply set $R'_t = L$ to increase the value of the contract. If $t \in (t_j, t_{j+1})$, the entrepreneur can apply the (t_j, t, ϵ) adjustment, until either $R'_t = L$ or $R'_{t_j} = 0$. In the former case, the value of the contract does not change; in the latter case, the value of the contract will increase, because all repayment after t_j become less risky. Note that in the adjustment process, the incentive compatibility is preserved. Hence, without loss of generality, we can consider the debt contract $\mathcal{R}$, in which at any $t \notin \mathcal{Q}$, $R_t = L$. The entrepreneur can then apply the (t_j, t_{j+1}, ϵ) adjustment for $t_j, t_{j+1} \in \mathcal{Q}$, such that the incentive-compatibility constraint is binding at each risky repayment date. (Here, we take $t_{N+1} = T$. It is possible that $t_{j+1} - t_j > K$, and so the incentive-compatibility constraint at t_{j+1} cannot be binding. However, in this case, the entrepreneur can just set $R'_{t_{j+1}} = L$ and $R'_{t_{j+1}+1} = K\Delta + L$ to increase the value of the contract.) This is guaranteed one-by-one from the last risky repayment date. In addition, the value of the contract does not change in the adjustment process. Therefore, below, we only need to show that a contract $\mathcal{R}$ with $R_s = L$ for all $s \notin \mathcal{Q}$ and $R_t = V_{t+1}$ for all $t \in \mathcal{Q}$ can be strictly improved.

We can construct a new contract $\mathcal{R}'$: $R'_s = R_s$ for all $s < t_1$ and $s > t_N + 1$, $R'_{t_1} = L$, $R'_{s+1} = R_s$ for all $s \in [t_1, t_N - 1]$, and $R'_{t_N+1} = (T - (t_N + 1))\,\Delta + L$. (If $t_N + 1 = T$, $R'_{t_N+1} = 0$. This case will be nested in the following proof.) Because $R_s = L$ for all $s \in [t_N + 2, T - 1]$ and $R_T = 0$, $V'_{t_N+2} = (T - (t_N + 1))\,\Delta + L$. Hence, the incentive-compatibility constraint is binding at $t_N + 1$. What's more, $V'_{t_N+1} = V_{t_N} = \Delta + L$. Therefore, at each previous date, the incentive-compatibility constraint is binding.

Hence, we just need to show that $\mathcal{D}(\mathcal{R}') > \mathcal{D}(\mathcal{R})$. Note that

$$
\begin{aligned}
&\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R})\\
\geq& R'_{t_1} + \frac{R'_{t_N+1}}{K^N} - \frac{R_{t_N} + R_{t_N+1}}{K^N}\\
=& L + \frac{(T - (t_N + 1))\,\Delta - ((T - t_N)\,\Delta + L)}{K^N}\\
=& L - \frac{\Delta + L}{K^N} > 0.
\end{aligned}
$$

Here, the first inequality is strict when $t_N + 1 = T$, and the last inequality is due to the definition of N^{**}.

Next, consider the case with $N < N^{**}$. Without loss of generality, we assume that $R_s = L$ for all $s < t_1$. The entrepreneur can construct the following new contract $\mathcal{R}'$: $R'_s = R_s$ for all $s < t_1 - K$, $R'_s = R_{s+K}$ for all $s \in [t_1 - K, T - 1 - K]$, $R'_{T-K} = K\Delta + L$, and $R'_s = L$ for all $s > T - K$. Because there are K periods after date $T - K$, R'_{T-K} is incentive compatible. In addition, because $R'_{T-K} = K\Delta + L$ and $R'_s = L$ for all $s > T - K$, $V'_{T-K} = \Delta + L$; hence, at any repayment date $s < T - K$, $\mathcal{R}'$ is incentive compatible because $\mathcal{R}$ is incentive compatible.

Note that $N < N^{**}$, and so $L < (\Delta + L)/K^{N+1}$. Therefore, we have

$$\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) = \frac{K(\Delta + L)}{K^{N+1}} - KL > 0.$$

Hence, the pledgeable income after adding the extra risky repayment increases. Therefore, $PI(N)$ is maximized at N^{**}.

Part 2: In the remainder of the proof, we assume $N \leq N^{**}$. When $N = 1$, to maximize the value of the debt, obviously the incentive-compatibility constraint must be binding at the risky repayment date t_1, and that at any date $s < t_1$, $R_s = L$. First, it is easy to see that $t_1 \geq T - K$; otherwise, because $R_s \leq L$ for all $s > t_1$, so

$$V_{t_1+1} \geq (T - t_1)\Delta + L > K\Delta + L \geq R_{t_1}$$

and incentive-compatibility constraint cannot possibly bind. For any $t_1 > T-K$, we can construct a new contract $\mathcal{R}'$: $R_s = L$ for all $s \neq T-K$, and $R'_{T-K} = K\Delta + L$. This is obviously incentive compatible. Now,

$$\begin{aligned}
&\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \\
=&\frac{K(\Delta+L)}{K} - \left[(t_1 - (T-K))L + \frac{(T-t_1)(\Delta+L)}{K}\right] \\
=&(K-(T-t_1))\left(\frac{\Delta+L}{K} - L\right).
\end{aligned}$$

Hence, $\mathcal{D}(\mathcal{R}')-\mathcal{D}(\mathcal{R}) \geq 0$, because $t_1 \geq T-K$ and $N \leq N^{**}$. Note that the inequality is strict if $t_1 > T-K$; hence, $PI(1)$ is uniquely attained by $R_{t_1} = K\Delta + L$ at date $t_1 = T-K$, and $R_s = L$ at date $s \neq t_1$. Then $PI(1) = (T-1-K)L + (\Delta + L)$.

Now, let's consider any $N \leq N^{**}$ and consider a contract $\mathcal{R}$ with $\#\mathcal{Q} = N$. Fixing all repayments at dates $s \geq t_2$, we have $V_{t_2+1} \geq R_{t_2}$ by the incentive compatibility of $\mathcal{R}$. Then, as the same argument of the case $N=1$, the value of the contract is maximized when $t_1 = t_2 - K$, $R_{t_1} = K\Delta + L$, $R_s = L$ for all $s < t_2$ but $s \neq t_1$. We can now increase the value of the contract by increasing R_{t_2} to V_{t_2+1}. Similarly, fixing all repayments at dates $s \geq t_n$ for all $n \leq N$, the value of the contract is maximized by setting $t_j = t_j - K$ for all $j < n$, $R_{t_j} = K\Delta + L$ at date t_j for all $j < n$, and $R_s = L$ at dates $s < t_n$ but $s \neq t_j$ for any $t_j < t_n$. Then, finally, because $R_T = V_{T+1} = 0$, the contract's value is maximized by setting $t_N = T-K$.

Therefore, to maximize the pledgeable income, the entrepreneur needs to make risky repayments at dates $t = T - iK$, where $i = 1, 2, \ldots, N$. In addition, at each risky repayment date, the risky repayment should be $K\Delta + L$, such that both the feasibility constraint and the incentive-compatibility constraint are binding; at other dates, the entrepreneur needs to repay L. Therefore, the maximum pledgeable income of a debt contract with N repayment dates is

$$PI(N) = (T-1-NK)L + \sum_{j=1}^{N} \frac{\Delta+L}{K^{j-1}}.$$

Part 3: For any $N \leq N^{**}$, if $D \in (PI(N-1), PI(N)]$, by the definition of $PI(N)$, the entrepreneur cannot use a contract with at most $N-1$ repayment dates to attain D, but she can use a contract with N repayment dates to attain D.

Now, we show that a contract with $N+p$ repayments, where $p \geq 1$, can be strictly improved. The first step is to show that, at all dates when the entrepreneur does not make risky repayments, the entrepreneur repays L. Suppose there is one date t such that $R_t < L$. If t is earlier than the first risky repayment t_1, the entrepreneur applies the $(t, t_{1,}\epsilon)$ adjustment by setting $R'_t = R_t + \epsilon$ and $R'_{t_1} = R_{t_1} - K\epsilon$. Then the creditor's participation constraint does not change, and the entrepreneur is at least as well off as before (strictly better if $R'_{t_1} \leq L$ after the adjustment). If $t \in (t_j, t_{j+1})$, then the entrepreneur can make the (t_j, t, ϵ) adjustment by setting $R'_{t_j} = R_{t_j} - \epsilon$ and $R'_t = R_t + \epsilon$. Such an adjustment will not change the incentive compatibility and the creditor's participation constraint either; again, the entrepreneur is at least not worse off. Hence, when we consider the optimal debt contract, the entrepreneur will repay L when she does not make risky repayments.The rest of the proof is the same as that of Claim 3 in the proof of Proposition **1** by iterated application of the (t_{N+p}, t_i, ϵ) adjustments.

As shown above, in the optimal debt contract, $t_{j+1} - t_j \leq K$ and $t_N \geq T-K$; otherwise, if some $t_j < t_{j+1} - K$ (we can denote by $t_{N+1} = T$), it is strictly better for the entrepreneur to set $R'_{t_j} = L$ and $R'_{t_j+1} = R_{t_j}$. Such a new contract is still incentive compatible, because the value between $t_j + 1$ and t_j will be greater than or equal to $K(\Delta + L)$. Therefore, $t_1 \geq T - NK$.

Part 4: For any $N \leq N^{**}$, if $D = PI(N)$, the optimal debt contract will have exactly N risky repayment dates. Then, by Part 2, to attain D by a contract with N risky repayment dates, the entrepreneur has to repay $K\Delta + L$ at dates $t_i = T - iK$ (for $i = 1, 2, \ldots, N$) and repay L at all other dates. Therefore, a unique optimal contract attains D, which is

$$R_t = \begin{cases} K\Delta + L, & \text{if } t \in \{T-K, T-2K, ..., T-NK\} \\ L & \text{otherwise.} \end{cases}$$

Proof of Lemma 2: Suppose D satisfies (8). Proposition 1 uniquely determines the set of repayment dates. Suppose ART is attained by some schedule $\mathcal{R}$ other than the one specified in the lemma. Then there must exist an $i \in [2, N-1]$ such that $R_{t_i} < K\Delta$, or $i = N$ and $R_{t_i} < j\Delta$. Given such an i, consider an alternative schedule $\mathcal{R}'$ given by a (t_1, t_i, ϵ) adjustment. It is clear that when $\epsilon < \frac{K\Delta - R_{t_i}}{K^N}$ (or $\frac{j\Delta - R_{t_N}}{K^N}$ if $i = N$), schedule $\mathcal{R}'$ is still incentive compatible. By Lemma 3, $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. The adjusted schedule $\mathcal{R}'$ increases ART:

$$ART(\mathcal{R}') - ART(\mathcal{R}) = \frac{\epsilon(t_i - t_1)}{KD} > 0.$$

Contradiction! So ART is uniquely attained by $R_{t_N} = j\Delta$, $R_{t_i} = K\Delta$ for all $i \in [2, N-1]$, and $R_{t_1} = K(D - \sum_{i=1}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N})$. It then follows from the definition of ART that

$$\begin{aligned}
& ART \\
= & \frac{1}{D}\left\{(D - \sum_{i=1}^{N-2}\frac{\Delta}{K^i} - \frac{j\Delta}{K^N})(T - j - (N-1)K)\right. \\
& \qquad \left. + \sum_{i=1}^{N-2}\frac{\Delta}{K^i}(T - j - (N-1-i)K) + \frac{j\Delta}{K^N}(T-j)\right\} \\
= & (T-j) - \frac{1}{D}\left[\sum_{i=0}^{N-2}\frac{\Delta}{K^{i-1}}(N-i-1) - (N-1)(K\Delta - R_{t_1})\right] \\
= & (T-j) - \frac{1}{D}\left[\sum_{i=0}^{N-2}\frac{\Delta}{K^{i-1}}(N-i-1) + K(N-1)\left(D - \sum_{i=0}^{N-2}\frac{\Delta}{K^i} - \frac{j\Delta}{K^N}\right)\right],
\end{aligned}$$

which is equivalent to Equation (14).

Proof of Prediction 1: Claim 3 in the proof of Proposition 1, together with the fact that $\Delta \sum_{i=0}^{N-1} \frac{1}{K^i}$ is decreasing in K, directly imply that $\#\mathcal{Q}$ is weakly increasing in K. It follows from Proposition 1 that if $t_i, t_{i+1} \in \mathcal{Q}$, then $t_{i+1} - t_i = K$. So, it is obvious that the time interval between two consecutive repayments is strictly increasing in K.

We now study ART as K increases to $K+1$. It follows from Equation (14) that ART can be rewritten as

$$ART = [T - j - (N-1)K] + \frac{1}{D}\left[\sum_{i=1}^{N-2}\frac{\Delta}{K^{i-1}}i + \frac{j\Delta}{K^{N-1}}(N-1)\right].$$

Obviously, if the increase in K does not change $\mathcal{Q}$, ART will decrease. Otherwise, suppose that the increase in K leads to a change of $\mathcal{Q}$. First, simple algebra shows that ART is decreasing in N and j, respectively.

Let N_K and j_K be the equilibrium outcome in Proposition 1 given K. Then, if $N_{K+1} \geq N_K$ and $j_{K+1} \geq j_K$, then $ART(K+1) < ART(K)$. Finally, we prove the result for $N_{K+1} > N_K$ and $1 \leq j_K - j_{K+1} \leq K - 1$:

$$\begin{aligned}
&ART(K+1) - ART(K) \\
\leq& K(N_K+1) - N_{K+1}(K+1) \\
&+ \frac{1}{D}\left[\sum_{i=1}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i-1}} i + \frac{j_{K+1}\Delta}{(K+1)^{N_{K+1}-1}}(N_{K+1}-1) - \sum_{i=1}^{N_K-2} \frac{\Delta}{K^{i-1}} i - \frac{j_K\Delta}{K^{N_K-1}}(N_K-1)\right].
\end{aligned}$$

Let's consider the terms in the bracket. First,

$$\begin{aligned}
&\frac{1}{D}\left[\sum_{i=1}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i-1}} i + \frac{j_{K+1}\Delta}{(K+1)^{N_{K+1}-1}}(N_{K+1}-1)\right] \\
=&(K+1)N_{K+1}\frac{1}{D}\left(\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^i} + \frac{(j_{K+1}-1)\Delta}{(K+1)^{N_{K+1}}}\right) \\
&\qquad - \frac{1}{D}(K+1)\left(\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^i}(N_{K+1}-i) - \frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}}}\right) \\
<&(K+1)N_{K+1} - \frac{1}{D}(K+1)\left(\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^i}(N_{K+1}-i) - \frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}}}\right).
\end{aligned}$$

Second,

$$\begin{aligned}
&-\frac{1}{D}\left[\sum_{i=1}^{N_K-2} \frac{\Delta}{K^{i-1}} i + \frac{j_K\Delta}{K^{N_K-1}}(N_K-1)\right] \\
=&-K(N_K+1)\frac{1}{D}\left(\sum_{i=0}^{N_K-2} \frac{\Delta}{K^i} + \frac{j_K\Delta}{K^{N_K}}\right) \\
&\qquad + \frac{1}{D}K\left(\sum_{i=0}^{N_K-2} \frac{\Delta}{K^i}(N_K+1-i) + \frac{2j_K\Delta}{K^{N_K}}\right) \\
\leq&-K(N_K+1) + \frac{1}{D}K\left(\sum_{i=0}^{N_K-2} \frac{\Delta}{K^i}(N_K+1-i) + \frac{2j_K\Delta}{K^{N_K}}\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&ART(K+1) - ART(K)\\
<&\frac{1}{D}\left[K\left(\sum_{i=0}^{N_K-2}\frac{\Delta}{K^i}(N_K+1-i)+\frac{2j_K\Delta}{K^{N_K}}\right)-(K+1)\left(\sum_{i=0}^{N_{K+1}-2}\frac{\Delta}{(K+1)^i}(N_{K+1}-i)-\frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}}}\right)\right]\\
\leq&\frac{1}{D}\left[K\left(\sum_{i=0}^{N_K-2}\frac{\Delta}{K^i}(N_{K+1}-i)+\frac{2j_K\Delta}{K^{N_K}}\right)-(K+1)\left(\sum_{i=0}^{N_{K+1}-2}\frac{\Delta}{(K+1)^i}(N_{K+1}-i)-\frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}}}\right)\right]\\
\leq&\frac{1}{D}\left[\sum_{i=0}^{N_K-2}\left(\frac{K\Delta}{K^i}-\frac{(K+1)\Delta}{(K+1)^i}\right)(N_{K+1}-i)-\sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-i)+\frac{2Kj_K\Delta}{K^{N_K}}+\frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}-1}}\right]\\
=&\frac{1}{D}\left[-N_{K+1}\Delta+\sum_{i=1}^{N_K-3}\left(\frac{\Delta}{K^i}-\frac{\Delta}{(K+1)^i}\right)(N_{K+1}-1-i)\right.\\
&\qquad\left.-\sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-i)+\frac{2Kj_K\Delta}{K^{N_K}}+\frac{(N_{K+1}-j_{K+1})\Delta}{(K+1)^{N_{K+1}-1}}\right]. \qquad \text{(A8)}
\end{aligned}
$$

Note that the second term in the bracket is 0 if $N_K \leq 3$; in such a case, simple mathematical induction can show that $ART(K+1) < ART(K)$. When $N_K \geq 4$, we have

$$
\begin{aligned}
&-N_{K+1}\Delta+\sum_{i=1}^{N_K-3}\left(\frac{\Delta}{K^i}-\frac{\Delta}{(K+1)^i}\right)(N_{K+1}-1-i)+\frac{2Kj_K\Delta}{K^{N_K}}+\frac{N_{K+1}\Delta}{(K+1)^{N_{K+1}-1}}\\
\leq&-N_{K+1}\Delta+(N_{K+1}-2)\sum_{i=1}^{\infty}\left(\frac{\Delta}{K^i}-\frac{\Delta}{(K+1)^i}\right)+\frac{2\Delta}{K^{N_K-2}}+\frac{N_{K+1}\Delta}{(K+1)^{N_{K+1}-1}}\\
=&\frac{N_{K+1}(1-(K-1)K)\Delta}{(K-1)K}-2\Delta\left[\frac{1}{(K-1)K}-\frac{1}{K^{N_K-2}}\right]+\frac{N_{K+1}\Delta}{(K+1)^{N_{K+1}-1}}\\
<&\frac{N_{K+1}(2-(K-1)K)\Delta}{(K-1)K}-2\Delta\left[\frac{1}{(K-1)K}-\frac{1}{K^{N_K-2}}\right]<0.
\end{aligned}
$$

This inequality, combined with the fact that all other terms in Equation (A8) are negative, implies that $ART(K+1) < ART(K)$. Hence, the conclusion holds for any $N_{K+1} > N_K$ and $j_{K+1} < j_K$. In all, ART decreases in K.

Proof of Prediction 2: Assume the variance of per-period cash flow is a constant: $\Delta^2(K-1)=\alpha^2$ for some constant $\alpha > 0$. Denote the solution by $\Delta_K = \frac{\alpha}{\sqrt{K-1}}$, which is decreasing in K. Therefore, $t_{i+1} - t_i = K$ is decreasing in Δ. In addition, $\Delta\sum_{i=0}^{N-1}\frac{1}{K^i} = \Delta\sum_{i=0}^{N-1}\frac{1}{\left(\frac{\alpha^2}{\Delta^2}+1\right)^i}$ is increasing in Δ, so Claim 3 in the proof of Proposition **1** implies that $\#\mathcal{Q}$ is decreasing in Δ.

In the remainder of the proof, we show that ART increases as Δ increases. We only consider the increase in Δ that decreases K to a smaller integer. Without loss of generality, we focus on the comparison between

$ART(\Delta_{K+1})$ and $ART(\Delta_K)$. Let's first consider the case that $\mathcal{Q}$ does not change. From (14),

$$ART = [T - j - (N-1)K] + \frac{\alpha}{D}\left[\sum_{i=1}^{N-2} \frac{1}{K^{i-1}\sqrt{K-1}} i + \frac{j}{K^{N-1}\sqrt{K-1}}(N-1)\right].$$

When Δ increases from Δ_{K+1} to Δ_K without changing $\mathcal{Q}$, ART increases, because ART is strictly decreasing in K. Similarly, if $N_{K+1} \geq N_K$ and $j_{K+1} \geq j_K$, ART also increases in Δ. Hence, we only need to show that the same property holds in the case that $j_K - j_{K+1} \leq K - 1$ and $N_{K+1} \geq N_K + 1$. Similar to the proof of Prediction 1, we have

$$\begin{aligned}
&ART(\Delta_{K+1}) - ART(\Delta_K)\\
\leq&\frac{1}{D}\left[K\left(\sum_{i=0}^{N_K-2}\frac{\Delta_K}{K^i}(N_{K+1}-i) + \frac{2j_K\Delta_K}{K^{N_K}}\right) - (K+1)\left(\sum_{i=0}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^i}(N_{K+1}-i) - \frac{(N_{K+1}-j_{K+1})\Delta_{K+1}}{(K+1)^{N_{K+1}}}\right)\right]\\
=&\frac{1}{D}\left[\sum_{i=0}^{N_K-2}\left(\frac{K\Delta_K}{K^i} - \frac{(K+1)\Delta_{K+1}}{(K+1)^i}\right)(N_{K+1}-i) - \sum_{i=N_K-1}^{N_{K+1}-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N_{K+1}-i) + \frac{2Kj_K\Delta_K}{K^{N_K}} + \frac{(N_{K+1}-j_{K+1})\Delta_{K+1}}{(K+1)^{N_{K+1}-1}}\right]\\
<&\frac{\alpha}{D}\left[N_{K+1}\left(\frac{K}{\sqrt{K-1}} - \frac{K+1}{\sqrt{K}}\right) + \sum_{i=0}^{\infty}\left(\frac{1}{K^i\sqrt{K-1}} - \frac{1}{(K+1)^i\sqrt{K}}\right)(N_{K+1}-1-i)\right.\\
&\left.\qquad - \sum_{i=N_K-1}^{N_{K+1}-2}\frac{1}{(K+1)^{i-1}\sqrt{K}}(N_{K+1}-i) + \frac{2Kj_K}{K^{N_K}\sqrt{K-1}} + \frac{(N_{K+1}-j_{K+1})}{(K+1)^{N_{K+1}-1}\sqrt{K}}\right]\\
<&\frac{\alpha}{D}\left[N_{K+1}\left(\frac{K}{\sqrt{K-1}} - \frac{K+1}{\sqrt{K}}\right) + (N_{K+1}-1)\left(\frac{1}{\sqrt{K-1}} - \frac{1}{\sqrt{K}}\right) + (N_{K+1}-2)\left(\frac{1}{(K-1)\sqrt{K-1}} - \frac{1}{K\sqrt{K}}\right)\right.\\
&\left.\qquad - \sum_{i=N_K-1}^{N_{K+1}-2}\frac{1}{(K+1)^{i-1}\sqrt{K}}(N_{K+1}-i) + \frac{2Kj_K}{K^{N_K}\sqrt{K-1}} + \frac{(N_{K+1}-j_{K+1})}{(K+1)^{N_{K+1}-1}\sqrt{K}}\right].
\end{aligned}$$

Similar to the proof of Prediction **1, we analyze the following terms in the bracket (when** $N_K \geq 4$**):**

$$\begin{aligned}
&N_{K+1}\left(\frac{K}{\sqrt{K-1}} - \frac{K+1}{\sqrt{K}}\right) + (N_{K+1}-1)\left(\frac{1}{\sqrt{K-1}} - \frac{1}{\sqrt{K}}\right) + (N_{K+1}-2)\left(\frac{1}{(K-1)\sqrt{K-1}} - \frac{1}{K\sqrt{K}}\right)\\
&\qquad + \frac{2Kj_K}{K^{N_K}\sqrt{K-1}} + \frac{N_{K+1}}{(K+1)^{N_{K+1}-1}\sqrt{K}}\\
<&N_{K+1}\left(\frac{K}{\sqrt{K-1}} - \frac{K+1}{\sqrt{K}} + \frac{1}{\sqrt{K-1}} - \frac{1}{\sqrt{K}} + \frac{1}{(K-1)\sqrt{K-1}} - \frac{1}{K\sqrt{K}}\right) - \left(\frac{1}{\sqrt{K-1}} - \frac{1}{\sqrt{K}}\right)\\
&\qquad - 2\left(\frac{1}{(K-1)\sqrt{K-1}} - \frac{1}{K\sqrt{K}} - \frac{1}{K^2\sqrt{K-1}}\right).
\end{aligned}$$

In the last inequality, the first term is negative when $K > 4$, the second term is obviously negative, and the third term is also less than 0. Note that the other two terms in the bracket are both negative. Therefore, under the condition that $K > 4$, $ART(\Delta_{K+1}) < ART(\Delta_K)$. This completes the proof.

Proof of Prediction 3: Claim 3 in the proof of Proposition **1** directly implies that $\#\mathcal{Q}$ is weakly increasing in D. In the rest of the proof, we study$ART(D)$ as D increases. There are two cases depending on whether D is at the boundary of (8).

Case 1: Both D and $D+\epsilon$ satisfy (8) for some common N and j. It then follows from Equation (14) that

$$ART(D+\epsilon) - ART(D) < 0,$$

because

$$\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}} (N-i-1) + K(N-1) \left(-\sum_{i=0}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N} \right) < 0.$$

Therefore, if the increase in D does not change $\mathcal{Q}$, $ART(D)$ is strictly decreasing in D.

Case 2: Suppose $D = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$ for some N and $j = 0, 1, ..., K-1$. When D marginally increases to $D+\epsilon$, the equilibrium debt contract then features $\#\mathcal{Q} = N$ and $t_N = T-(j+1)$. Therefore,

$$\begin{aligned}
&\lim_{\epsilon\to 0} ART(D+\epsilon) \\
=&\lim_{\epsilon\to 0} (T-(j+1)) - \frac{1}{D+\epsilon} \left[\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}} (N-i-1) + K(N-1) \left((D+\epsilon) - \sum_{i=0}^{N-2} \frac{\Delta}{K^i} - \frac{(j+1)\Delta}{K^N} \right) \right] \\
=&ART(D) - 1 + \frac{\frac{\Delta}{K^N}(N-1)K}{D} < ART(D),
\end{aligned}$$

because $\frac{\frac{\Delta}{K^N}(N-1)K}{D} < 1$. Therefore, we conclude that $ART(D)$ is strictly decreasing in D if the marginal change of D does not change $\mathcal{Q}$; however, when the marginal increase in D leads to a different set of repayment dates, ART has a discrete drop.

Proof of Proposition 5: Consider a repayment schedule with $R_t > K\Delta$ for some $t \in \mathcal{Q}$. Let R_{t_j} be the last repayment with $R_{t_j} > K\Delta$. There are two cases. First, the repayment schedule has exactly j repayments, and so R_{t_j} is also the last repayment. Then

$$T - t_j = \left\lceil \frac{R_{t_j}}{\Delta} \right\rceil > K;$$

otherwise, $R_{t_j} > V_{t_j+1}$, violating the incentive-compatibility constraint. Then, fixing all previous $j-1$ repayments (the time and the amount), the entrepreneur may consider the following adjustment: $R'_{t_j} = K\Delta$ and $R'_{t_j+K} = R_{t_j} - K\Delta$. Such a new repayment schedule is still incentive compatible; otherwise, if $R'_{t_j+K} > V_{t_j+K+1} = (T-t_j-K)\Delta$, $R_{t_j} > (T-t_j)\Delta = V_{t_j+1}$, violating the assumption that the original repayment schedule is incentive compatible.

Because the first $j-1$ repayments do not change, if the entrepreneur can repay R_{t_j} at date t_j, she is able to make the repayments R'_{t_j} and R'_{t_j+K} (because of saving). Indeed, there is a positive probability that the entrepreneur cannot repay R_{t_j} but can make the repayments R'_{t_j} and R'_{t_j+K}, because the project may generate positive cash flows between t_j+1 and t_j+K. In addition, because of saving, the creditor's participation constraint is also satisfied. Therefore, the entrepreneur can even reduce R'_{t_j+K} to a certain R''_{t_j+K} and still keep the creditor's participation constraint satisfied.

Now, let's compare V_{t_j} and V'_{t_j}. Denote by S_t the total funds the entrepreneur can use to make repayment at date t. We can calculate

$$V_{t_j} = \Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R_{t_j} + (T-t_j)\Delta\right)$$

and

$$
\begin{aligned}
&V'_{t_j}\\
=&\Delta + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\Big\{-R'_{t_j} + K\Delta\\
&\qquad + \Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S'_{t_j} \geq R'_{t_j}\right)\left(-R''_{t_j+K} + (T - t_j - K)\Delta\right)\Big\}\\
=&\Delta + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\left(-R'_{t_j} + K\Delta\right)\\
&\qquad + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S'_{t_j} \geq R'_{t_j}\right)\left(-R''_{t_j+K} + (T - t_j - K)\Delta\right)
\end{aligned}
$$

As we argued above, the same first $j-1$ repayments imply that $S_{t_j} = S'_{t_j}$, and then because of $R_{t_j} = R'_{t_j} + R'_{t_j+K} > R'_{t_j}$, if $\Pr\left(S_{t_j} \geq R'_{t_j}\right) > 0$,

$$
\Pr\left(S_{t_j} \geq R'_{t_j}\right) \geq \Pr\left(S_{t_j} \geq R_{t_j}\right).
$$

In addition, if $S_{t_j} \geq R_{t_j}$, then $S_{t_j} - R'_{t_j} = R'_{t_j+K} > R''_{t_j+K}$, and so

$$
\Pr\left(S_{t_j} \geq R'_{t_j}\right)\Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S_{t_j} \geq R'_{t_j}\right) > \Pr\left(S_{t_j} \geq R_{t_j}\right).
$$

These imply that

$$
\begin{aligned}
&V'_{t_j}\\
>&\Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R'_{t_j} + K\Delta\right) + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R''_{t_j+K} + (T - t_j - K)\Delta\right)\\
>&\Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R'_{t_j} + K\Delta - R'_{t_j+K} + (T - t_j - K)\Delta\right)\\
=&V_{t_j}.
\end{aligned}
$$

Hence, the adjustment makes the entrepreneur strictly better off.

In the second case, the repayment schedule has more than j repayments. Then, by assumption, all repayments after date t_j are at most $K\Delta$. The entrepreneur can then make all the repayments after date t_j as late as possible, until the point that delaying one of these repayments by one date violates the incentive-compatibility constraint. Then the entrepreneur can iteratedly apply the (t_{i-1}, t_i, ϵ) adjustment for all $i \geq j$ beginning from $i = N$ (the last repayment), such that the incentive-compatibility constraint is binding at each repayment date after date t_j. The entrepreneur will not be worse off from this adjustment, by the same logic as in the first case. Suppose now that R_{t_j} is still strictly greater than $K\Delta$. Then $V_{t_{j+1}} = (t_{j+1} - t_j)\Delta$, and $t_{j+1} - t_j = \left\lceil \frac{R_{t_j}}{\Delta} \right\rceil > K$. Hence, the same argument in the first case will prove that such a contract is not optimal for the entrepreneur.

Appendix B. Numerical Solution

This appendix describes the numerical procedure used to calculate the results in Section 3.3, where we present optimal debt structures when the period cash flow X_t follows an IID lognormal distribution.

We calculate the optimal debt structure using numerical finite-horizon dynamic programming methods. Recall that, written recursively, the value function at the beginning of period t is given by

$$V_t = E(X_t) + \Pr(X_t \geq R_t)\left(-R_t + V_{t+1}\right).$$

The optimal debt structure $\mathcal{R} = \{R_t\}$ is then the solution to the following maximization problem:

$$\begin{aligned} & \max\nolimits_{\mathcal{R}} V_1 & \\ \text{s.t.} \quad & R_t \leq V_{t+1} & \text{(IC)} \\ & \mathcal{D}(\mathcal{R}) = D & \text{(IR)}, \end{aligned} \tag{B1}$$

where $\mathcal{D}(\mathcal{R})$ denotes the expected payoff to creditors,

$$\mathcal{D}(\mathcal{R}) = \sum_{t=1}^{T} \prod_{s \leq t} \Pr(X_s \geq R_s) R_t, \tag{B2}$$

and the parameter D is the amount of outside financing that needs to be raised. We assume that the period cash flow is IID lognormally distributed,

$$X_t \sim \text{Lognormal}(-\frac{\sigma^2}{2}, \sigma^2),$$

where $-\frac{\sigma^2}{2}$ and σ are the mean and the volatility parameters of the underlying normal distribution, so that the expected period cash flow is normalized to $E(X_t) = 1$.

In describing the numerical algorithm, it is useful to define the amount of financing raised by promised repayments scheduled from date t onward as

$$D_t = \sum_{k=t}^{T} \prod_{s \leq k} \Pr(X_s \geq R_s) R_k.$$

The algorithm we use to solve for the optimal repayment profile is essentially backward induction.

1. Grid points: We first solve for the single-period pledgeability-maximizing face value

   $$\overline{R} = \arg\max_{R} P(X_t \geq R) R.$$

   We then discretize the interval $[0, \overline{R} + 0.1]$ at increments of 0.1, rounding up $\overline{R} + 0.1$ to the closest decimal point. In total, depending on the parametrization, there are around $N_R = 20$ grid points for R. We discretize the value of debt D using $N_D = 1,000,000$ to $10,000,000$ points on $[0, D_{max}]$, where D_{max} is picked to be slightly above the firm's borrowing capacity (i.e., maximum pledgeable income). To achieve better computational efficiency, we choose the lowest possible D_{max} that does not impose a binding constraint on the optimization problem.

The computational challenge is that the grid points on D must be much finer than the grid points on R in order to guarantee the precision of the algorithm. For example, when the date-10 face value R_{10} changes by one grid of 0.1, its impact on the value of debt D_1 is scaled by the survival probabilities given several positive earlier repayments. Hence, for the algorithm to detect variations in all R_t, the number of grid points for D must be orders of magnitude larger than the number of grid points for R. This issue becomes more severe with more potential repayment dates, and thereby it acts as a constraint on the number of dates T for which we can numerically calculate the optimal debt structure. Section 3.3 presents results for an 11-period model ($T = 11$). For these results, we have checked that the discretization is fine enough. In particular, the results are unchanged when we move to even finer levels of discretization in D.

2. Given $T = 11$ periods in total, we have the following initial conditions:

$$D_{11}^* = 0$$

(no financing can be raised by promising repayments only at date 11) and

$$V_{11}^*(D_{11}) = \begin{cases} E(X_{11}) = \Delta & \text{if } D_{11} \leq 0 \\ 0 & \text{if } D_{11} > 0 \end{cases}.$$

3. Given $V_{t+1}^*(D_{t+1})$, we then solve $R_t^*(D_t)$ and $V_t^*(D_t)$ using a standard recursive formulation:

   (a) For a given D_t, for any R_t in the grid, we solve for $\hat{D}_{t+1}$ such that

   $$D_t = P(X_t \geq R_t)(R_t + \hat{D}_{t+1}).$$

   $\hat{D}_{t+1}$ represents the required value of repayments from date $t+1$ onward in order to raise D_t at date t. If $\hat{D}_{t+1} > D_{max}$, then the repayment profile is infeasible and can be discarded because, under this repayment profile, the firm cannot possibly raise D_t.

   (b) We then verify that the IC holds:

   $$R_t \leq V_{t+1}^*(\hat{D}_{t+1}).$$

   If the IC does not hold, then this R_t should not be considered a candidate for the following maximization problem.

   (c) We then choose R_t such that equity value V_t is maximized:

   $$R_t^* = \arg\max_{R_t} E(X_t) + P(X_t \geq R_t)(-R_t + V_{t+1}^*(\hat{D}_{t+1})),$$

   where the $\hat{D}_{t+1}$ is the continuation value of debt calculated in (a) and (b). Finally, using the optimal R_t^*, we calculate

$$V_t^*(D_t) = E(X_t) + P(X_t \geq R_t^*)(-R_t^* + V_{t+1}^*(\hat{D}_{t+1})),$$

completing the iteration for period t.

4. For any initial value of $D = D_1$, we calculate the stream of R_t^* for $t = 1, 2, ..., 10$ (recall that R_{11} is necessarily zero).